JOHN STACHEL

THE FIRST TWO ACTS

PROLOGUE: THE DEVELOPMENT OF GENERAL RELATIVITY, A DRAMA IN THREE ACTS

In 1920, Einstein wrote a short list of "my most important scientific ideas".[1] The final three items on the list are:

1907 Basic idea for the general theory of relativity

1912 Recognition of the non-Euclidean nature of the metric and its physical determination by gravitation

1915 Field equations of gravitation. Explanation of the perihelion motion of Mercury.

Einstein's words provide the warrant for comparing the development of general relativity to a three-act drama:

Act I (1907) The formulation of the "basic idea," to which he soon referred as the equivalence principle.

Act II (1912) The mathematical representation of the gravitational field by a symmetric second rank tensor field, which enters into the line element of a four-dimensional spacetime; hence this tensor is usually referred to as the (pseudo-)metric of spacetime.[2]

Act III (1915) The formulation of the now-standard Einstein field equations for the metric field, and use of its spherically-symmetric solution to explain the anomalous precession of the perihelion of Mercury.

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1 Einstein Archives, Hebrew University of Jerusalem, Control Index No. 11 196. (Hereafter, only the number of such items will be cited.) It appears from Einstein to Robert Lawson, 22 April 1920 (1–010), that it was written for the biographical note in (Einstein 1920a). This is the English translation of the fifth German edition of (Einstein 1920b). I thank Dr. Josef Illy of the Einstein Papers for locating Einstein's letter to Lawson. In (CPAE 8), it is incorrectly calendared under 1917 (see pp. 1005–1006).

2 Properly speaking, the term "metric" should be restricted to line elements with positive-definite signature; those with an indefinite signature are more properly termed "pseudo-metrics." But Einstein, and following him most physicists, referred to the four-dimensional tensor field with Minkowski signature as the metric tensor, and I shall follow that usage.

Jürgen Renn (ed.). *The Genesis of General Relativity,* Vol. 1
Einstein's Zurich Notebook: *Introduction and Source*.

Act III certainly does not represent the end of general relativity, but a certain point of closure in its development, signaled by Einstein in his 1916 review paper (Einstein 1916a):

> "According to the general theory of relativity, gravitation plays an exceptional role as opposed to the other forces, in particular the electromagnetic..." (p. 779);

consequently, he concluded his exposition with a "Theory of the Gravitational Field" (pp. 801–822).

Up to this point, the story had been essentially an account of Einstein's struggles.[3] Now that the final form of the gravitational field equations had been achieved and one of its predictions validated, the theory became the property of the physics and astronomy communities. Its further development and interpretation became a subject of discussion among many participants, among whom Einstein's voice did not always carry the day.[4]

This book tells the story from the opening curtain of Act I in 1907 until the curtain goes down on Act III at then end of 1915. The great bulk of it is devoted to Act III, embracing the events between 1912 and 1915. In particular, it centers on the understanding of the Zurich Notebook, which opens Act III and has made a signal contribution to our understanding of subsequent events. As in most plays, the final act contains the *dénouement*; but the crucial events that lead up to it take place in the first two acts. It was his formulation of the equivalence principle in Act I, and constant adherence to it as the guiding thread in his search for a relativistic theory of gravitation that set Einstein apart from other physicists who were working on the problem of fitting gravitation within the framework of the (special) theory of relativity. And as usual, the high point of the drama comes in Act II, with Einstein's remarkable decision to represent gravitation, not by a scalar field, but by the ten components of a tensor field that also describes the chronogeometry of a non-flat four-dimensional spacetime.

It is at this point, with an expression for the line element in terms of the metric tensor field, that the Zurich Notebook opens. Clearly, it cannot be fully understood or evaluated without prior knowledge of what happened during the first two acts. Unfortunately, no equivalent of the Zurich Notebook has been found for this period from 1907–1912. So this chapter attempts to present what can be learned—or surmised—about what happened on the basis of Einstein's published papers and correspondence, as well as his later reminiscences.[5] In keeping with the documentary character of this book, rather than attempting to summarize them, I shall often cite Einstein's words *in extenso*.

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3 David Hilbert played a significant role in the final moments of Act III, although not the one that is often attributed to him. See "Hilbert's Foundation of Physics: From a Theory of Everything to a Constituent of General Relativity" (in vol. 4 of this series).

4 For Einstein's side of this discussion during its first years, see (CPAE 7).

5 For another account, see (Pais 1982, Part IV, 177–296). For some critical comments, see (Stachel 1982); reprinted in (Stachel 2002, 551–554).

ACT I: THE EQUIVALENCE PRINCIPLE: "THE MOST FORTUNATE THOUGHT OF MY LIFE"

In 1920,[6] Einstein recalled how he first arrived at the ideas behind the equivalence principle:

> While I was occupied (in 1907) with a comprehensive survey of the special theory[7] for the "Yearbook for Radioactivity and Electronics," I also had to attempt to modify Newton's theory of gravitation in such a way that its laws fitted into the theory. Attempts along these lines showed the feasibility of this enterprise, but did not satisfy me, because they had to be based on physical hypotheses that were not well-founded. Then there came to me the most fortunate thought of my life in the following form:
>
> Like the electric field generated by electromagnetic induction, ... the gravitational field only has a relative existence. *Because, for an observer freely falling from the roof of a house, during his fall there exists*—at least in his immediate neighborhood—*no gravitation field*. Indeed, if the observer lets go of any objects, relative to him they remain in a state of rest or uniform motion, independently of their particular chemical or physical composition [note by AE: air resistance is naturally ignored in this argument]. The observer is thus justified in interpreting his state as being at rest.
>
> Through these considerations, the unusually extraordinary experimental law, that all bodies fall with equal acceleration in the same gravitational field, immediately obtains a deep physical significance. For if there were just one single thing that fell differently from the others in the gravitational field, then with its help the observer could recognize that he was falling in a gravitational field. If such a thing does not exist—which experiment has shown with great precision—then there is no objective basis for the observer to regard himself as falling in a gravitational field. Rather, he has the right to regard his state as one of rest and, with respect to a gravitational field, his neighborhood as field free. The experimental fact of the material-independence of the acceleration due to gravity is thus a powerful argument for the extension of the relativity postulate to coordinate systems in non-uniform relative motion with respect to each other .... The generalization of the relativity principle thus indicates a speculative path towards the investigation of the properties of the gravitational field (pp. 24–25).

Einstein alludes here to his initial attempts to set up a special-relativistic theory of gravitation, but gives no details. In 1933 he gave the fullest account of how he "arrived at the equivalence principle by a detour [*Umweg*]"[8] through such attempts.[9]

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6 "Grundgedanken und Methoden der Relativitätstheorie, in ihrer Entwicklung dargestellt," in (CPAE 7, Doc. 31).

7 I shall follow the common but anachronistic practice of referring to "the special theory of relativity." During this period, Einstein initially called it "the principle of relativity" [*das Relativitätsprinzip*] and then, following the practice of others, "the theory of relativity" [*die Relativitätstheorie*]. For details, see the discussion in the Editorial Headnote, "Einstein on the Theory of Relativity," (CPAE 2, 254); reprinted in (Stachel 2002, 192).

8 "Erinnerungen-Souvenirs" (Einstein 1955, 145–153) was reprinted as "Autobiographische Skizze," in (Seelig 1955, 9–17). Citation from "Autobiographische Skizze," p. 14.

9 "Einiges über die Entstehung der allgemeinen Relativitätstheorie," the German text of a lecture given at the University of Glasgow, 20 June 1933. The German text was published in (Einstein 1934, 248–256). Cited from the paperback edition edited by Carl Seelig: (Seelig 1981, 134–138).

After mentioning his doubts after 1905 about the privileged dynamical role of inertial systems, and his early fascination by Mach's idea that the acceleration of a body is not absolute, but relative to the rest of the bodies in the universe, he turns to the events of 1907:

> I first came a step closer to the solution of the problem when I attempted to treat the law of gravitation within the framework of special relativity. Like most authors at the time, I attempted to establish a field law for gravitation, since the introduction of an unmediated action at a distance was no longer possible, at least in any sort of natural way, on account of the abolition of the concept of absolute simultaneity.
>
> The simplest thing naturally was to preserve the Laplacian scalar gravitational potential and to supplement Poisson's equation in the obvious way by a term involving time derivatives, so that the special theory of relativity was satisfactorily taken into account. The equation of motion of a particle also had to be modified to accord with the special theory. The way to do so was less uniquely prescribed, since the inertial mass of a body might well depend on its gravitational potential. This was even to be expected on the basis of the law of the inertia of energy.
>
> However, such investigations led to a result that made me highly suspicious. For according to classical mechanics, the vertical acceleration of a body in a vertical gravitational field is independent of the horizontal component of its velocity. This is connected with the fact that the vertical acceleration of a mechanical system, or rather of its center of mass, in such a gravitational field turns out to be independent of its internal kinetic energy. According to the theory I was pursuing, however, such an independence of the gravitational acceleration from the horizontal velocity, or from the internal energy of a system, did not occur.[10]
>
> This did not accord with an old fact of experience, that all bodies experience the same acceleration in a gravitational field. This law, which can also be formulated as the law of equality of inertial and gravitational mass, now appeared to me in its deep significance. I was most highly amazed by it and guessed that in it must lie the key to the deeper understanding of inertia and gravitation (pp. 135–136).

Turning from later reminiscences, let us see how Einstein presented his approach to gravitation in 1907:[11]

> Up to now we have only applied the principle of relativity, i.e., the presupposition that the laws of nature are independent of the state of motion of the reference system, to *acceleration-free* reference systems. Is it conceivable that the principle of relativity also holds for systems that are accelerated relative to each other?
>
> This is not the place for an exhaustive treatment of this question. Since, however, it is bound to occur to anyone who has followed the previous applications of the relativity principle, I shall not avoid taking a position on the question here.

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10 As Einstein later realized, a special-relativistic theory of gravitation that does justice to the equivalence principle is possible, and indeed one was developed a little later by Gunnar Nordström. For an account of Nordström's theories, and Einstein's reaction to them, see (Norton 1992).

11 "Über das Relativitätsprinzip und die aus demselbem gezogenen Folgerungen," (Einstein 1907); "Berichtigungen," in (Einstein 1908).

> Consider two systems in motion Σ_1 and Σ_2. Let Σ_1 be accelerated in the direction of its X-axis, and let γ be the magnitude (constant in time) of this acceleration. Let Σ_2 be at rest, but in a homogeneous gravitational field that imparts an acceleration $-\gamma$ in the direction of the X-axis to all objects. As far as we know, the laws of physics with respect to Σ_1 do not differ from those with respect to Σ_2; this is due to the circumstance that all bodies in a gravitational field are equally accelerated. So we have no basis in the current state of our experience for the assumption that the systems Σ_1 and Σ_2 differ from each other in any respect; and therefore in what follows shall assume the complete physical equivalence of a gravitational field and the corresponding acceleration of a reference system.
>
> This assumption extends the principle of relativity to the case of uniformly-accelerated translational motion of the reference system. The heuristic value of this assumption lies in the circumstance that it allows the replacement of a homogeneous gravitational field by a uniformly accelerated reference system, which to a certain extent is amenable to theoretical treatment.[12]

Some further comments on this equivalence in his next paper on gravitation in 1911[13] are illuminating. He notes that in both systems, objects subject to no other forces fall with constant acceleration:

> For the accelerated system K' [corresponding to the 1907 Σ_1–JS], this follows directly from the Galileian principle [of inertia–JS]; for the system K at rest in a homogeneous gravitational field [corresponding to the 1907 Σ_2–JS], however, it follows from the experimental fact that in such a field all bodies are equally strongly uniformly accelerated. This experience of the equal falling of all bodies in a gravitational field is the most universal with which the observation of nature has provided us; in spite of that, this law has not found any place in the foundations of our physical picture of the world. ... From this standpoint one can as little speak of the *absolute acceleration* of a reference system, as one can of the *absolute velocity* of a system according to the usual [special–JS] theory of relativity. [note by AE: Naturally, one cannot replace an *arbitrary* gravitational field by a state of motion of the system without a gravitational field; just as little as one can transform all points of an arbitrarily moving medium to rest by a relativity transformation.] From this standpoint the equal falling of all bodies in a gravitational field is obvious.
>
> As long as we confine ourselves to purely mechanical processes within the realm of validity of Newtonian mechanics, we are certain of the equivalence of the systems K and K'. Our point of view will only have a deeper significance, however, if the systems K and K' are equivalent with respect to all physical processes, i.e., if the laws of nature with respect to K agree completely with those with respect to K'. By assuming this, we obtain a principle that, if it really is correct, possesses a great heuristic significance. For by means of theoretical consideration of processes that take place relative to a uniformly accelerated reference system, we obtain conclusions about the course of processes in a homogeneous gravitational field. (CPAE 3, 487–488)

With hindsight, one can see that Einstein's attempt to find the best way to implement mathematically the physical insights about gravitation incorporated in the equivalence principle was hampered significantly by the absence of the appropriate mathematical concepts. His insight, as he put is a few years later, that gravitation and inertia are "essentially the same" [*wesensgleich*],[14] cries out for implementation by

12 See (CPAE 7, 476). Also see p. 495 for a discussion of the meaning of uniform acceleration.
13 "Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes," (Einstein 1911b).

their incorporation into a single inertio-gravitational field, represented mathematically by a non-flat affine connection on a four-dimensional manifold. But the concept of such a connection was only developed *after*, and largely in response to, the formulation of the general theory. So Einstein had to make do with what was available: Riemannian geometry and the tensor calculus as developed by the turn of the century, i.e., based on the concept of the metric tensor, without a geometrical interpretation of the covariant derivative. As I have suggested elsewhere, this absence is largely responsible for the almost three-year lapse between the end of Act I and the close of the play.[15]

ACT II: THE METRIC TENSOR: "JUST WHAT ARE COORDINATES ACTUALLY SUPPOSED TO MEAN IN PHYSICS?"

In 1949, Einstein himself raised the question of what was responsible for this long delay:

> This [recognition that the relativity principle had to be extended to non-linear transformations–JS] took place in 1908. Why were a further seven years required for setting up the general theory of relativity? The principal reason is that one does not free oneself so easily from the conception that an immediate physical significance must be attributed to the coordinates.[16]

Both the question and answer thus concern the entire period between 1907 (or 1908) and 1915. In 1933 Einstein made the answer more precise, and confined it to a shorter period of time:

> I soon saw that, according the point of view about non-linear transformations required by the equivalence principle, the simple physical interpretation of the coordinates had to be abandoned; i.e., one could no longer require that coordinate differences be interpreted as signifying the immediate results of measurements with ideal measuring rods and clocks. This recognition tormented me a great deal because for a long time I was not able to see just what *are* coordinates actually supposed to mean in physics? The resolution of this dilemma was reached around 1912. (Seelig 1981, 137)[17]

Einstein reference to 1912 is a clear allusion to his introduction of the metric tensor. But, as his reference to "a further seven years" after 1908 in the previous quotation suggests, the problem of the meaning of coordinates in general relativity was by no

14 In 1912 Einstein regarded "the equivalence of inertial and gravitational mass" as being reducible to the "essential likeness [*Wesensgleichheit*] of both of these elementary qualities of matter and energy"; and asserted that his theory of "the static gravitational field" allows him to regard it "as physically the same in essence [*wesensgleich*] as an acceleration of the reference system." See (Einstein 1912c, 1063).

15 The entire complex of problems raised in this paragraph is discussed at length in "The Story of Newstein or: Is Gravity just Another Pretty Force?" (in vol. 4 of this series).

16 From Albert Einstein's "Autobiographical Notes," which, although published first in 1949 (Einstein 1949, 2–94), were actually written in 1947. Cited here from (Einstein 1979, 63).

17 See (Seelig 1981, 1).

means completely resolved with the introduction of the metric tensor. Only with the resolution in 1915 of the "hole argument" [*Lochbetrachtung*] against general covariance that Einstein developed in 1913, did Einstein fully solve this problem; but discussion of the post-1912 aspects of the question will be found in later chapters.[18]

Now let me return to the problem of coordinates as Einstein saw it in 1907–1908. It is worth emphasizing that Einstein attributed his success in formulating the special theory in 1905 in no small measure to his insistence on physically defining coordinate systems that allow one to attach direct physical significance to coordinate differences:

> The theory to be developed—like every other electrodynamics—rests upon on the kinematics of rigid bodies, since the assertions of each such theory concern relations between rigid bodies (coordinate systems), clocks and electromagnetic processes. Taking this into account insufficiently is the root of the difficulties, with which the electrodynamics of moving bodies currently has to contend.[19]

Little wonder that Einstein was "tormented" by the problem of "just what coordinates are actually supposed to mean in physics" once they lose their direct physical significance!

This problem arose in the course of the application of the equivalence principle to linearly accelerated frames of reference and the attempt to apply it to uniformly rotating frames, both considered within the confines of Minkowski space.[20] Its resolution came out of Einstein's work on a theory of the static gravitational field, in particular on the equations of motion of a particle in this field; and his attempt to generalize this static theory to non-static fields.

Both problem areas, accelerated systems of reference in Minkowski space and static gravitational fields, ultimately led Einstein beyond the confines of Minkowski space to the consideration of non-flat Riemannian spacetimes. For convenience of exposition, I shall discuss these two strands of the story as if they were the subject of two separate scenes of Act Two, culminating in a third scene that ends the act. While it is broadly true that events in Scene One precede those in Scene Two, and certainly true that they all precede the events in Scene Three, to the extent that this division suggests a strict chronological separation between events in the First and Second Scenes, it does a certain violence to the actual course of events. However, it seems preferable to run this risk rather than attempt to jump back and forth between events in each of the intertwined strands of the story.[21]

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18 For a historical discussion of Einstein's hole argument, see my 1980 Jena paper, published as (Stachel 1989); and reprinted in (Stachel 2002, 301–337).

19 "Zur Elektrodynamik bewegter Körper," (Einstein 1905); reprinted in (CPAE 2, 276–306), citation from p. 277. For further discussion of this paper see the Editorial Note "Einstein on the Theory of Relativity," in (CPAE 2, 253–274); reprinted in (Stachel 2002, 233–244)

20 For a discussion of the development of Einstein's concept of the equivalence principle, see (Norton 1985); reprinted in (Howard and Stachel 1989, 5–47).

21 Since contemporary documents are cited with dates, the chronological sequence can easily be reconstructed.

SCENE I: "TO INTERPRET ROTATION AS REST":

I shall start by again citing Einstein's 1949 comments on coordinates:

> The change [from the viewpoint that coordinates must have an immediate metrical significance] came about in more-or-less the following way.
>
> We start with an empty, field-free space as it appears with respect to an inertial system in accord with the special theory of relativity, as the simplest of all conceivable physical situations. Now if we imagine a non-inertial system introduced in such a way that the new system (described in three-dimensional language) is uniformly accelerated in a (suitably defined) direction with respect to the inertial system; then, with respect to this system, there exists a static parallel gravitational field. In this case, the reference system may be chosen as a rigid one, in which three-dimensional Euclidean metric relations hold. But that time [coordinate–JS], in which the field appears static, is *not* measured by *equally constituted* clocks at rest [in that system–JS]. From this special example, one already recognizes that, when one allows non-linear transformations of any sort, the immediate metrical significance of the coordinates is lost. One *must* introduce such transformations, however, if one wants to justify the equality of gravitational and inertial mass by the foundations of the theory, and if one wants to overcome Mach's paradox concerning inertial systems.[22]

Examination of Einstein's 1907 paper[23] shows that this account correctly reflects its contents. Einstein first demonstrates that—at least to first order in the acceleration—the spatial coordinates in a uniformly accelerating frame of reference retain their direct physical significance in terms of measuring rods; and thus, by the principle of equivalence, they still do so in the equivalent gravitational field. He then goes on to show that what he calls "the local time σ" [he uses both "*Ortszeit*" and "*Lokalzeit*" as names], which is essentially the proper time as measured by an ideal clock at a fixed point of the frame, differs from the "time τ," which he later called the "universal" [*universelle*] time,[24] which must be used to define simultaneity of distant events if one wants a time coordinate expressing the static nature of the gravitational field that is equivalent to the uniformly-accelerated one.

Thus, by the end of 1907, Einstein knew that differences between the "universal" time coordinates of events in a uniform gravitational field do not correspond to differences in the readings of ideal clocks in that field. It is true that he had shown that, at least to first order in the field strength, spatial coordinate differences still correspond to the results of measurements with rigid rods. But the fact that he felt compelled to demonstrate this for uniform gravitational fields suggests that he anticipated the pos-

22 "Autobiographical Notes," (Einstein 1979, 62 and 64); see note 16. An idea of what he meant by "Mach's paradox concerning inertial systems" may be gathered from the citations of Mach in Einstein's article, "Ernst Mach," (Einstein 1916b). See also (Einstein 1916a); reprinted in (CPAE 6, 284–339, Section 2), "Über die Gründe, welche eine Erweiterung des Relativitätspostulates naheliegen," pp. 286–288.

23 "Über das Relativitätsprinzip," (CPAE 2, Section 18, 476–480); for the full references, see note 11.

24 Einstein did not actually introduce this term until 1912, in his first paper on the static gravitational field, in which he contrasts the "local time" and the "universal time" (for the full reference, see note 36).

sibility that similar problems might arise for the spatial coordinates in more complicated gravitational fields.

Einstein did not publish anything on gravitation between 1908 and 1911, but he continued to think about the subject:

> Between 1909–1912 while I had to teach theoretical physics at the Zurich and Prague Universities I pondered ceaselessly on the problem.[25]

The earliest surviving indication that Einstein contemplated an extension of the relativity principle beyond linearly accelerated systems dates from 1909:

> The treatment of the uniformly rotating rigid body seems to me to be of great importance on account of an extension of the relativity principle to uniformly rotating systems along lines of thought analogous to those that I attempted to carry out for uniformly accelerated translation in the last section of my paper published in the *Zeitschrift für Radioaktivität*.[26]

What he had in mind is made more explicit in 1912 in a letter to his friend Michele Besso. After a rather full account of his new static theory (to be discussed below), he concludes: "You see that I am still far from being able to interpret [*auffassen*] rotation as rest. Every step is devilishly difficult..."[27]

As his reference to a "uniformly rotating rigid body" suggests, a solution to the problem of "interpreting rotation as rest" seemed to him to depend on developing a theory of rigid bodies in special relativity. In 1910 he wrote of this

> child of sorrow [*Schmerzenskind*], the rigid body. ... one should attempt to devise hypotheses about the behavior of rigid bodies that would permit a uniform rotation.[28]

Born had provided a definition of a relativistic rigid body in 1909, but he only discussed the case of linearly accelerated motion in any detail.[29] Further clarification soon came:

> The latest relativity-theoretical investigations of Born and Herglotz interest me very much. It really seems that in the theory of relativity there does not exist a "rigid" body with 6 degrees of freedom.[30]

This was disturbing, but brought new hope: If rigid *bodies* are incompatible with the special theory, rigid *motions* are not. In 1911, Laue summarized the situation concisely:

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25 "Autobiographische Skizze," (Seelig 1955, 14).

26 Einstein to Arnold Sommerfeld, 29 September 1909, (CPAE 5, 210). Einstein incorrectly names the title of the journal in which his earlier paper was published (see Einstein 1907). Einstein had described the main theme of the last section of this paper in an earlier letter to Sommerfeld, Einstein to Arnold Sommerfeld, 5 January 1908, (CPAE 5, 86).

27 Einstein to Michele Besso, 26 March 1912,(CPAE 5, 435–438); citation from p. 436.

28 Einstein to Arnold Sommerfeld,19 January 1910, (CPAE 5, 228–230); citation from p. 229.

29 See (Born 1909a). It was his report on this work at the 1909 Salzburg meeting of the *Versammlung deutscher Naturforscher und Ärzte*, (Born 1909b), that provoked the above-cited letter of 1909 from Einstein to Sommerfeld. See also (Born 1910), discussed below.

30 Einstein to Jakob Laub, 16 March 1910, (CPAE 5, 231–233); citation from p. 232.

> The limiting concept of a body that is rigid under all circumstances, which is so useful everywhere in classical mechanics, in my opinion cannot be taken over [to the special theory–JS] on account of the impossibility of indefinitely large velocities for the propagation of elastic deformations. However this does not exclude a body moving at times like a rigid one; even according to classical mechanics, under certain circumstances a drop of fluid can move as if it were rigid. (Laue 1911, note at bottom of p. 107)

In short, the problem had been transformed from a dynamical one (what is a relativistic "rigid body" and how does it behave when accelerated?) to a kinematical one (given Born's relativistic definition, what types of "rigid motion" are possible?). Progress on the kinematical problem was much easier.[31] Indeed, in the paper cited by Einstein, Herglotz showed:

> that, as soon as one of the points of a [rigid–JS] body in Mr. Born's sense is fixed, it can only rotate uniformly about an axis passing through this point, like the usual rigid body. (Herglotz 1910, 403)

In the course of classifying all solutions of Born's rigidity condition, Herglotz gave the explicit form of the solution for rigid rotation about the z-axis (Herglotz 1910, 412). So Einstein could discuss the kinematics of such an ideal rigid rotation, and the gravitational field which is equivalent to the inertial forces in such a rotating frame, without having to solve the dynamical problem of what types of physical system could actually undergo such a motion.

But for Einstein, there remained a second, Machian type of question: What distribution of matter could *induce* the gravitational field in a frame at rest that is equivalent to the inertial forces in an accelerated frame? Einstein first considered this question for linear acceleration, so we shall discuss it before returning to the problem of rotation. In a 1912 paper on gravitational induction,[32] Einstein showed that, as a consequence of the inertia of energy and the equivalence principle, a spherical shell of matter K accelerated linearly relative to an unaccelerated frame exerts such an inductive accelerating (gravitational) effect on a particle P enclosed in the shell.[33] He also showed that:

31 Of course, a dynamical problem remained: how to create the circumstances that would lead a non-rigid body of a particular constitution to execute a particular rigid motion. But such special dynamical problems could be attacked *after* the general kinematical problem was solved.

32 "Gibt es eine Gravitationswirkung, die der elektrodynamischen Induktionswirkung analog ist?"(Einstein 1912a); reprinted in (CPAE 4, 175–179). Einstein published the article in this journal because it was part of a *Festschrift* for his friend, Heinrich Zangger, an expert in forensic medicine.

33 Presumably, this would be Einstein's answer to Laue's objection to the equivalence principle: "For the gravitational field in the system K [at rest, with a uniform gravitational field] there must be present a body that causes gravitation, not however for the accelerated system K'. So a search for it must immediately decide whether there is a real gravitational field or only an accelerated reference system," Max Laue to Albert Einstein, 27 December 1911, (CPAE 5, 384). This letter is discussed further below. Einstein evidently tried to answer Laue's objection in a footnote to his next paper, submitted two months later: "The masses that produce this field must be thought of as at infinity" (Einstein 1912b, 356). The paper on gravitational induction followed almost immediately.

> the presence of the massive shell K increases the inertial mass of the particle P within it. This makes likely the assumption that the entire inertia of a massive particle is an effect of the presence of all the other masses, based on a sort of interaction with the latter. This is completely the same standpoint that E. Mach had upheld in his acute investigations on the subject (*E. Mach, The Development of the Principles of Dynamics. Chapter Two. Newton's Views on Time, Space and Motion*).[34] How far this conception is justified will be seen when we are in the happy possession of a usable dynamics of gravitation (p. 177).

Presumably, Einstein already had in mind the application of this induction idea to rotational acceleration. In 1921, discussing the development of the general theory, Einstein wrote:[35]

> Can gravitation and inertia be identical [*wesensgleich*]? The posing of this question leads directly to the General Theory of Relativity. Is it not possible for me to regard the earth as free from rotation, if I conceive of the centrifugal force, which acts on all bodies at rest relative to the earth, as being a "real" field of gravitation (or part of such a field)? If this idea can be carried out, then we shall have proved in very truth the identity of gravitation and inertia. For the same effect [*Wirkung*] that is regarded as *inertia* from the point of view of the system not taking part in the rotation can be interpreted as *gravitation* when considered with respect to the system that shares the rotation.

I believe that the phrase "or part of such a field" makes clear what Einstein had in mind as his ultimate goal. The total gravitational field of the earth in a frame in which it is at rest (i.e., a co-rotating frame) consists of two parts: a gravito-static term, which would be present even if the earth were not in rotation (this is the Newtonian gravitational field), and a gravito-stationary term. The latter is usually interpreted as an inertial field, consisting of centrifugal and Coriolis terms, which would exist even if the earth were massless, i.e., in *any* rotating frame of reference. But, in accord with the principle of equivalence, these terms may be interpreted as a gravito-stationary field in a non-rotating frame of reference.

In summary: Because of his attraction to Mach's program from the beginning of his search for a theory of gravitation based on the equivalence principle, the aim of interpreting rotation as rest-plus-a-gravitational-field appears to have loomed large in Einstein's motivation. This motive led him to consider uniformly rotating systems of reference soon after his 1907 treatment of uniformly linearly accelerated systems. But only after the clarification of the question of rigid motions do we find any signs of progress on the rotation problem.

The study of uniformly rotating reference systems then led him to the conclusion that, in this case, the spatial coordinates cannot be given a direct physical meaning. He announced this result in February 1912, in an uncharacteristically tentative tone, in the course of a discussion of the spatial coordinates in a linearly accelerated frame of reference K:[36]

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34 Exactly the same words about Mach's book occur in "Einstein's Scratch Notebook," reproduced with transcription in (CPAE 3, "Appendix A," 564–596; see p. 592).

35 "A Brief Outline of the Development of the Theory of Relativity, (Einstein 1921, 783). A German draft, "Kurze Skizze zur Entwicklung der Relativitätstheorie," has been used to correct the English text. Both appear in (CPAE 7).

> The spatial measurement of K is done with measuring rods that—when compared with each other at rest at the same place in K— possess the same length; the theorems of [Euclidean–JS] geometry are assumed to hold for lengths measured in this way, and thus also for the relations between the coordinates x,y,z and other lengths. That this stipulation is allowed is not obvious; rather it contains physical assumptions that eventually could prove incorrect. For example, it is highly probable that they do not hold in a uniformly rotating system, in which, on account of the Lorentz contraction, the ratio of the circumference to the diameter, using our definition of lengths, must be different from π.

There is evidence suggesting that he had this rotating disk argument before 1912, but it is indirect and suggestive rather than conclusive. Einstein's letter of 1909 to Sommerfeld, cited above,[37] was written just a few days after the Salzburg meeting of the Society of German Natural Scientists and Physicians [*Deutsche Naturforscher und Ärzte*], at which Einstein had spoken. As the editors of the Einstein Papers note:[38]

> At the Salzburg meeting Max Born had presented a paper on rigid body motion in special relativity ..., on which Sommerfeld had commented in the discussion following the paper. Einstein and Born had discussed the subject and had discovered that setting a rigid disk into rotation would give rise to a paradox: the rim becomes Lorentz-contracted, whereas the radius remains invariant (see Born 1910, p. 233).[39] The existence of this paradox was first pointed out in print by Paul Ehrenfest (1880–1933) in a paper that was received on the date of this letter.[40]

While the line of argument about setting a rigid disk into rotation (which has come to be called "Ehrenfest's Paradox") is not the same as that in Einstein's treatment of an already rigidly-rotating disk,[41] the basic idea in both arguments is the same: Relative to an inertial frame, measuring rods at rest in a uniformly rotating frame of reference *do not* contract if aligned in a radial direction, but *do* contract if aligned orthogonally to a radial direction.

So it is reasonable to suppose that Einstein, already alerted to the possibility that coordinate differences in an accelerating frame might not be directly interpretable in terms of physical measurements and having read Herglotz's 1910 paper, realized that this was indeed the case for the spatial coordinates in a rigidly rotating frame of reference.[42]

Indeed there is evidence that, by the end of 1911 at the latest, Einstein saw an analogy between the gravitational field that, according to the equivalence principle (conceiving rotation as rest), is equivalent to the inertial field in a uniformly rotating

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36 "Lichtgeschwindigkeit und Statik des Gravitationsfeldes," (Einstein 1912b); reprinted in (CPAE 4, 130–145); citation from p. 131. (This is his first paper on the static gravitational field, discussed at greater length below.) For a translation of a longer portion of this passage and a fuller discussion of the rotating disk problem, see (Stachel 1980); reprinted in (Howard and Stachel 1989, 48–62) and in (Stachel 2002, 245–260).

37 See note 28.

38 See (CPAE 5, 211, n. [5]).

39 The reference reads: "Mr. P. Ehrenfest ... showed in a very simple way that a body at rest can never be brought into uniform rotation; I had already discussed the same fact with Mr. A. Einstein in Salzburg."

40 "Gleichförmige Rotation starrer Körper und Relativitätstheorie," (Ehrenfest 1909).

41 For Einstein's way of avoiding Ehrenfest's paradox, see (Stachel 1980, 6–7 and 9).

frame of reference and the magnetostatic (or electro-stationary) field due to a stationary circular current distribution.[43] In the letter cited earlier,[44] Max Laue alludes to:

> your [i.e., Einstein's] question whether the gravitational field strength should be represented by a four-vector or a six-vector.

We shall return to this letter at some length below, but for the moment, consider the implications of Einstein's question. The most natural generalization of the Newtonian gravitational field strength would be a four-vector, since the force exerted by the Newtonian field depends only on the position and not the velocity of a mass in that field. On the other hand, the electric and magnetic field strengths together constitute a six-vector, and the force exerted on a charge in an electromagnetic field depends on the position (electric force) and velocity (magnetic force) of the charge. Around the turn of the century, H. A. Lorentz had suggested a gravitational theory modeled on electromagnetism, in which there were gravitational analogues of the electric and magnetic forces (Lorentz 1899–1900a).[45]

Thus, Einstein's question to Laue suggests that, by the end of 1911, he had reason to believe that the force exerted by the most general gravitational field might also be velocity dependent. In his 1911 paper, he had considered the gravitational analogue of a constant electrostatic field, and he was soon to consider the analogue of the general electrostatic field.[46] By 18 February 1912, Einstein was already communicating some of his results.[47]

There is also evidence that, by February 1912, Einstein was already considering the gravitational analogue of a magnetostatic field. Paul Ehrenfest visited Einstein in Prague during the last week of February, and Ehrenfest's diary entry for 24 February 1912 contains the following lines:

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42 Curiously, neither he nor any other contemporary ever refers to a 1910 paper by Theodor Kaluza (of later five-dimensional Kaluza-Klein theory fame) solving the problem of the "proper geometry" [*Eigengeometrie*] of a Born rigidly-rotating body (Kaluza 1910). Kaluza was prevented by illness from presenting his work at the 1910 Königsberg meeting of the *Deutsche Naturforscher und Ärzte* which may help to explain its lack of impact on the rigid body discussion.

43 However, there is also an important difference between the two: The gravitational field equivalent to the inertial forces in an accelerated reference frame does not appear to correspond to any material sources, while the analogous electromagnetic fields are produced by a charged ring—rotating or not. As noted above, this was the purport of Max Laue's criticism of Einstein's treatment of the gravitational field equivalent to the inertial forces in a uniformly accelerated frame of reference (see note 33), to which Einstein hoped to provide a Machian answer.

44 Max Laue to Albert Einstein, 22 December 1911, (CPAE 5, 384–385); citation from p. 385.

45 English translation "Considerations on Gravitation," in (Lorentz 1899–1900b).

46 This was probably at least in part in response to Abraham's work on the problem of gravitation, which appeared early in 1912. Abraham's first two papers on the subject, dated "December 1911," were received 14 December and published in the issue of 1 January 1912 of *Physikalische Zeitschrift* (Abraham 1912a; Abraham 1912b). There is evidence that Einstein had corresponded with Abraham about his theory before publication (see below).

47 See Einstein to Hendrik Antoon Lorentz, (CPAE 5, 411–413); reference to gravitation on p. 413.

> Einstein told me about his gravitational work. [I omit some equations referring to the static case] Centrifuging of radiation.[48]

A subsequent letter from Ehrenfest makes clear the meaning of the final, rather cryptic phrase. He reports that a Russian colleague, Michael Frank, has "put me in a very uncomfortable situation" by asking Ehrenfest to translate into German a work concerning "the geometry of light rays in a uniformly rotating laboratory." After describing Frank's work, Ehrenfest adds:

> If one wanted to transform the acceleration field of uniform rotation into a corresponding force field at rest, as you do in your paper 'On the Influence of Gravitation ...' for uniform linear acceleration, then this substitute force-field would also have to give the proper Coriolis deflection for light rays.—That is the content of [Frank's] note. The thing is embarrassing [*peinlich*] for me since you had already communicated this argument to me. ... I told him that you had already told me about this (I remembered it naturally just at the moment when "Coriolis" was recognizable.)[49]

So "centrifuging of radiation" refers to "the geometry of light rays in a uniformly rotating laboratory," a problem on which Einstein had evidently worked before 24 February 1912 when he presented his results to Ehrenfest. Einstein's mention of "Coriolis" indicates that he had in mind a velocity-dependent gravitational force. It was presumably the publication of Frank's paper[50] that decided Einstein against publishing his own version of the results on light rays. He wrote Ehrenfest:

> Translate that work [of Frank-JS] in tranquility. I do not arrogate to myself any relativity-monopoly! Everything that is good is also welcome. You needn't send the proofs.[51]

So by February 1912, as seems probable on the basis of his own writings; and surely by April, on the basis of Ehrenfest's letter, Einstein was aware that the gravitational field equivalent to a rotating frame of reference would have to exert a force on a light ray that depends not only on its position but on (at least the direction of) its velocity—something that is incompatible with a scalar theory of gravitation. So, even while writing his first paper on the static gravitational field, he was aware that a scalar theory was not possible for more general gravitational fields. I shall return to this point below.

Ehrenfest continued to work on the kinematic aspects of the gravito-stationary problem. In a postcard, he writes that he has solved the "Problem: To determine the *most general* field of world-lines that is equivalent to a *stationary* gravitational field." He mentions two "special cases": "hyperbolic motion" (i.e., constant linear acceleration) and "uniform rotation."[52] Subsequent letters outline the proof his solution.[53] In

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48 Rijksmuseum voor de Geschiednis der Natuurwetenschappen, Leiden: Ehrenfest Collection, Ehrenfest Notebook 4–11, Microfilm number 12.

49 Ehrenfest to Einstein, draft letter before 3 April 1912, (CPAE 5, 439–445), citation from p. 440.

50 "Bemerkung betreffs der Lichtausbreitung in Kraftfeldern," (Frank 1912). The paper is dated 7 March 1912. Frank does not suggest that the force field equivalent to a force-free rotating frame of reference could be gravitational.

51 Einstein to Ehrenfest, 25 April 1912, (CPAE 5, 450–451); citation from p. 450.

reply to one of these letters, Einstein says that he does not understand Ehrenfest's result,[54] but adds some comments on the problem:

> A rotating ring does not generate a static field in this sense [the sense of Papers I and II, see note 70–JS], although it is a time-independent field. In such a field the reversibility of light paths does not hold.[55] My case corresponds to the electrostatic field in electromagnetic theory, while the more general static case would also include the analogue of the static magnetic field. I am not yet that far along. The equations I have found only relate to the static case of masses at rest.[56]

I have now given reasons for believing that, at the earliest by 1909 and the latest by the end of 1911, Einstein was aware of problems with the interpretation of both temporal and spatial coordinates in accelerating frame of reference in Minkowski space; and that, by February 1912 at the latest, he had every reason to expect that the full theory of gravitation would have to pass beyond the limits of a four-dimensional scalar theory, on the one hand; and, at least spatially, beyond the limits of Euclidean geometry. Now I shall turn to Einstein's investigation of static gravitational fields, which ultimately led to the resolution of these problems.

SCENE II: "THE SPEED OF LIGHT IS NO LONGER CONSTANT"

It may well have been Max Laue who directed Einstein's attention to the crucial importance of the gravitational potential, and the possibility of its replacement by the variable speed of light. The letter Laue sent Einstein at the end of 1911 was cited above, but I must now quote from it at greater length. (Unfortunately we do not have Einstein's letter, if there was one, to which this is a reply.)[57] Discussing Einstein's 1911 paper,[58] he writes:

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52 Ehrenfest to Einstein, 14 May 1912, (CPAE 5, 460–461). In later chapters, these two "special cases" will become very familiar to the reader since they are the two test cases that Einstein uses again and again to evaluate candidate gravitational field equations.

53 Only Ehrenfest's drafts of his letters have been preserved: Ehrenfest to Einstein: after 16 May 1912, (CPAE 5, 461–464, see 462–463); 29 June 1912, (CPAE 5, 487–496). Einstein to Ehrenfest, 25 April 1912, (CPAE 5, 450–451, see 451); 27 April 1912, (CPAE 5, 455); before 20 June 1912, (CPAE 5, 484–486, see 485–486). (Only letters containing references to gravitation are cited.) Ehrenfest later published a paper on this subject: "On Einstein's Theory of the Stationary Gravitation Field," (Ehrenfest 1913a); original version, (Ehrenfest 1913b).

54 Einstein to Ehrenfest, before 20 June 1912, (CPAE 5, 484–486, see p. 485).

55 This assertion is analogous to the result discussed above that Frank had published, but with a subtle difference. Like Einstein, Frank had discussed Minkowski spacetime as seen from a uniformly rotating frame of reference and the field (he does not specify it as gravitational) equivalent to the inertial forces present in such a frame. Here, Einstein is discussing the gravitational field generated by a rotating material ring, which he must have realized would be non-Minkowskian since this is true even for the field of a non-rotating material ring.

56 Einstein to Ehrenfest, before 20 June 1912, (CPAE 5, 484–486, see p. 486).

57 Since Einstein does not raise the question of whether the gravitational field strength is a four-vector or a six vector in his 1911 paper, I assume that the phrase "Your question" in Laue's letter refers to either a previous letter or conversation.

> It seems extraordinarily characteristic to me that the gravitational potential thereby acquires a physical significance, which is completely lacking for the electrostatic potential. One could, in principle, immediately determine the former by measurement of the velocity of light.

This comment may well have been the cue that prompted Einstein's replacement of the gravitational potential by the variable speed of light in his gravito-static theory.

But, as indicated in the last section, the letter contains more significant clues to the direction in which Einstein was heading. Laue continues:

> Your question, whether the gravitational field strength should be represented by a four-vector or a six-vector, is thereby settled. Not it [the field strength–JS] but rather the potential accordingly seems to me to be the primary concept, the four-dimensional representation of which must be investigated.[59]

A little background is helpful in assessing the full significance of this comment of Laue's. After initially slighting the significance of Minkowski's four-dimensional reformulation of the special theory, Einstein had started to study it in earnest around 1910,[60] probably at least in part in response to Sommerfeld's exposition of a four-dimensional vector algebra and analysis (Sommerfeld 1910a; Sommerfeld 1910b),[61] and its incorporation and further development in Laue's textbook—the first on special relativity.[62]

A further motive for this study was Einstein's decision to include the four-dimensional approach in a major review article that he agreed to write in 1911, and started work on by 1912.[63] In this review he notes that an anti-symmetric second rank tensor "is, following Sommerfeld, usually designated as a six-vector" [*Sechservektor*],[64]

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58 See (Einstein 1911b).

59 Max Laue to Albert Einstein, 27 December 1911, (CPAE 5, 384).

60 In 1908, Einstein and Laub thought it worthwhile to publish a paper rederiving Minkowski's four-dimensional results on electrodynamics (see Minkowski 1908) in three-dimensional form because that "work makes rather great demands mathematically on the reader," see (Einstein and Laub 1908). In a review talk on special relativity given in January 1911, Einstein included a brief discussion of "the highly interesting mathematical development that the theory has undergone, primarily due to Minkowski who unfortunately died so young," noting that it had led to "a very perspicacious representation of the theory, which essentially simplifies its application" see (Einstein 1911a). For a discussion of Minkowski's work and the varying forms of its assimilation by the physics and mathematics communities, see (Walter 1999).

61 He states that the formalism he presents "is (aside from imaginary coordinates) an immediate generalization of the customary three-dimensional vector methods", and provides "a complete substitute for the matrix calculus used by Minkowski" (Sommerfeld 1910a, 749).

62 Laue notes that he has "taken into account extensively the mathematical development of the theory that Sommerfeld has recently given" (Laue 1911, vi). Einstein commented: "His book on relativity theory is a little masterpiece," Einstein to Alfred Kleiner, 3 April 1912, (CPAE 5, 445–446).

63 This article, prepared for Erich Marx's *Handbuch der Radiologie*, was completed but has been published only recently: See "Manuscript on Relativity," (CPAE 4, 9–108). For its history, see the Editorial Note, "Einstein's Manuscript on the Special Theory of Relativity," (CPAE 4, 3–8).

64 See (CPAE 4, 72). Sommerfeld had given the names "four-vector" and "six-vector" to what Hermann Minkowski had called "spacetime vectors of type I and II," respectively (Minkowski 1908, 65–68).

and notes that the electromagnetic field strength, the components of which are the electric (see p. 9) and magnetic (see p. 10) field strengths, is a six-vector (see p. 81).

So it is reasonable to assume that Einstein had already mastered the four-dimensional formalism by the time that he raised the question of whether the gravitational field strength is a six-vector (as in the electromagnetic case) or a four-vector (as would be the case of it were the gradient of a scalar field). Laue's reply assumes knowledge of the fact that the electromagnetic six-vector is the curl of the electromagnetic potential four-vector;[65] which breaks up into the electric potential (a three-scalar) and the magnetic (three-)vector potential with respect to any inertial frame. If we look only at the electrostatic field strength, it can be written as the three-gradient of the electric potential. But, Laue points out, the electrostatic potential is of no physical significance (because of the possibility of what are now called gauge transformations of the electromagnetic potentials);[66] while Einstein's 1911 work showed that the gravitational potential has an immediate physical significance because of its influence on the speed of light. Therefore, Laue suggests, the important question is: What is the four-dimensional representation of the gravitational *potential*? The fact that, in the static case, it reduces to a single quantity that behaves as a scalar under three-dimensional spatial transformations is *not* decisive for answering this question. The same is true of the electrostatic potential; yet the latter is known to be the fourth (i.e., timelike) component of a four-vector.

Thus, Laue's comment could have served to draw Einstein's attention away from the representation of the gravitational field strength, and toward the question that, within a few months, was to occupy him: What is the four-dimensional representation of the gravitational potential in the non-static case?

Laue's comment, like all of his and Sommerfeld's work on the four-dimensional formalism, is situated within the context of the special theory of relativity. But, as we have seen, with Einstein's interpretation of the equivalence principle as implying an enlargement of the relativity group, Einstein had already moved beyond that context; and he soon moved into the context of non-flat spacetimes. Indeed, there is a comment by Einstein himself dating from mid-1912, on the question of the four-versus six-vector representation of the gravitational field strength, that suggests the need for this shift of context:

> If the gravitational field can be interpreted within our present [i.e., special–JS] theory of relativity [*sich ... im Sinne unserer heutigen Relativitätstheorie deuten läßt*], then this can only happen in two ways. One can consider [*auffassen*] the gravitational vector either as a four-vector or as a six-vector. [In either case] one arrives at results that contradict the ... consequences of the law of the gravitational mass of energy, [namely] ... that gravitation acts more strongly on a moving body than on the same body in case it is at rest. ... It must

65 See Laue's book, *Das Relativitätsprinzip*, (Laue 1911, 99–100), which defines the "four-potential vector" [*Viererpotential*] as the four-vector, the four-curl of which is the electromagnetic field six-vector; and notes that the four-potential vector is only determined up to the four-gradient of a scalar.

66 This was, of course, long before discussions of the physical significance of the electromagnetic potentials, based on the Aharonov-Bohm effect, took place.

> be a task of the immediate future to create a relativistic-theoretical schema in which the equivalence of gravitational and inertial mass finds expression.[67]

In a similar vein, he wrote to Wien that:

> the [special–JS] relativity theory imperatively demands a further development since the gravitational vector cannot be fitted into the relativity theory with constant c if one demands the *gravitational* mass of energy[68]

As it turned out, such a theory would not only pass beyond the bounds of the special theory; it would involve spacetimes with non-flat line elements, as we shall soon see.

To summarize: There are good reasons to suggest that, by the beginning of 1912, Einstein already realized that he would ultimately have to go beyond a scalar theory of gravitation. His strategy was to proceed in a step-by-step fashion towards a full dynamical theory. The first step in the program was to consider what I have called above the gravito-static case, the gravitational analogue of electrostatics; but he was already thinking about the next step, the gravito-stationary case, the gravitational analogue of magnetostatics. His ultimate goal was to develop a theory for time-dependent gravitational fields.

Let us look at the first step, gravito-statics. By March 1912 he was able to write Paul Ehrenfest:

> The investigations of gravitational statics (point mechanics electromagnetism gravito-statics) are complete and satisfy me very much. I really believe that I have found a part of the truth. Now I am considering the dynamical case, again also proceeding from the more special to the more general [case–JS].[69]

Einstein was referring to his two papers on the static gravitational field (hereafter cited as "Paper I" and "Paper II"), completed in February and March 1912 respectively.[70] These papers center on the gravitational potential, as Laue had suggested, but effect a crucial transformation of the problem in line with Laue's comment that the gravitational potential could "in principle be determined by measurement of the speed of light." In his 1911 paper, Einstein had already shown that, with a certain definition of the universal time:[71]

> in a static gravitational field a relation between c [the speed of light–JS] and the gravitational potential exists, or in other words, that the field is determined by c. (Einstein 1912b, 360)

In Papers I and II $c(x,y,z)$, the spatially variable but temporally constant and direction-independent speed of light, completely replaces the gravitational potential.

67 "Relativität und Gravitation. Erwiderung auf eine Bemerkung von M. Abraham," (Einstein 1912c). The paper was received on 4 July 1912. The citations are from pp. 1062–1063.

68 Einstein to Wilhelm Wien, 17 May 1912, (CPAE 5, 465).

69 Einstein to Paul Ehrenfest, 10 March 1912, (CPAE 5, 428).

70 Paper I:"Lichtgeschwindigkeit und Statik des Gravitationsfeldes," (Einstein 1912b); reprinted in (CPAE 4, 130–145); Paper II: "Zur Theorie des statischen Gravitationsfeldes," (Einstein 1912d); reprinted in (CPAE 4, 147–164). For a discussion of these papers, see the Editorial Note "Einstein on Gravitation and Relativity: The Static Field," (CPAE 4, 122–128).

Most of Paper I is concerned with establishing the gravitational field equation that c obeys, and the equations of motion of a (test) particle in the static gravitational field described by c. Much of Paper II is concerned with a revision of the field equation of Paper I (Section 4), and a crucial mathematical reformulation of the equations of motion (The “Supplement” to the Proofs).[72]

Einstein’s introduction of a variable speed of light brought down much scorn upon him at the time,[73] but it was absolutely crucial in initiating the sequence of steps that lead to the culmination of Act II: Einstein’s leap from a scalar to a tensorial gravitational potential, in which $c(x,y,z)$ becomes one of the ten components of the metric tensor used to construct the line element of a non-flat spacetime.

Paper I also contains another step in the process. Einstein shows that, if one uses a light clock, for example, to measure:

> the local time, which Abraham denotes by l, then this stands to the universal time $[t]$ in the relation $dl = c\ dt$. (Einstein 1912b, 366)

In retrospect (remembering that here c is non-constant), we recognize in this equation the relation between the differential element of the proper time dl between two events at the same place (i.e., $x,y,z =$ const) in a static field, and the coordinate differential dt between the times of the two events, using the preferred static time coordinate t. This equation begins to answer the question of the relation between coordinates and physical measurements in a gravitational field that had been puzzling Einstein for almost five years.

But before expanding on this point, let me turn to a further step in the process, contained in Paper II. The equations of motion of a particle in a static gravitational field, developed in Paper I, were rewritten in Lagrangian form in a “Supplement to

71 “The time in the field [that is] defined by the stipulation that the speed of light c depends indeed upon the position but not on direction” as Einstein explained to Michele Besso, 26 March 1912, (CPAE 5, 435). This definition of the time is given in more detail in Paper I: “We think of the time in the [uniformly accelerated–JS] system K as measured by clocks of such a nature and such a fixed arrangement at the spatial points of K that the time intervals—measured with them—that a light ray takes to go from a point A to a point B of the system K does not depend on the moment of emission of the light ray at A [static condition]. Further it turns out that simultaneity can be defined without contradiction in such a way that, with respect to the settings of the clocks, the stipulation is satisfied that all light rays passing a point A of K have the same speed of propagation, independent of their direction” [isostropy condition] (Einstein 1912b, 357–358).

72 The revision of the gravitational field equation is discussed in some detail in “Pathways out of Classical Physics ...” (in this volume). The remainder of Paper II is concerned with electromagnetism (Sections 1 and 2) and thermodynamics (Section 3) in a static gravitational field, topics I shall not consider.

73 Abraham, for example, exulted: “Einstein ... had already given up his postulate of constancy of the velocity of light at the turn of the year, which was so essential for his earlier theory; in a recent work he abandons the requirement of the invariance of the equations of motion under Lorentz transformations, thereby delivering the *coup de grace* to the theory of relativity. Those who, like the author, have repeatedly had to warn against the siren song of this theory, can only greet with satisfaction the fact that its originator has now convinced himself of its untenability” (Abraham 1912e).

the Proofs" of Part II. This step proved to be so significant that it soon led to the final resolution of the problem of the correct representation of the gravitational potentials.

> It is noteworthy that the equations of motion of a particle [*materielle Punkt*] in the gravitational field take a very simple form when they are given the form of Lagrange's equations. Namely, if one takes
>
> $$H = -m\sqrt{c^2 - q^2},$$
>
> then ... For a particle moving in a static gravitational field without the action of external forces, there holds accordingly
>
> $$\delta\left\{\int H\, dt\right\} = 0,$$
>
> or
>
> $$\delta\left\{\int \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}\right\} = 0.$$
>
> Here too—as was proved by Planck for the usual [i.e., special] theory of relativity—it is seen that the equations of analytical mechanics possess a significance that extends far beyond Newtonian mechanics. Hamilton's equation as finally written down lets us anticipate [*ahnen*] the structure of the equations of motion of a particle in a dynamical gravitational field. (Einstein 1912d), 458)

Einstein's lecture notes on mechanics, which he had been teaching since 1909,[74] show his familiarity with the use of variational techniques to derive the Lagrangian equations of motion (CPAE 3, 91–95, 116–117). Even more important for present purposes, he stressed the coordinate-invariant nature of the resulting equations of motion:

> The Cartesian coordinates of the particle no longer enter into the [variational–JS] principle. It holds independently of whatever coordinates we use to determine the position of the particles of the system (p. 93).

Now we have in hand all the strands, the interweaving of which finally allowed Einstein to take the great leap forward to a metric theory of gravitation. But let me emphasize that, however much they may help us in retrospect to understand the process, there remains something almost uncanny in how Einstein made the choices that led him so far from the path trodden by other physicists in the search for a relativistic theory of gravitation. He was about to enter an entirely new land, in which the spacetime structure becomes a dynamical field.

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74 "Lecture Notes for Introductory Course on Mechanics at the University of Zurich, Winter Semester 1909–1910," (CPAE 3, 11–129). See also the Editorial Note, "Einstein's Lecture Notes," (CPAE 3, 3–10, especially Section II, 4–6). Einstein also taught mechanics in Prague during the winter semester of 1911; see "Appendix B: Einstein's Academic Courses," (CPAE 3, 598–600).

SCENE III: "TEN SPACETIME FUNCTIONS":

Just what did the variational reformulation of the equations of motion lead Einstein to "anticipate"? I propose to answer this question with the help of the first three sections of his following paper on the topic, the "*Entwurf*" paper, published early in 1913.[75] After introductory comments discussing the equivalence principle, Section 1 treats "The equations of motion of a particle in a static gravitational field." Except for one small but significant detail, it is just an expanded version of the "Supplement." The detail is notational: he writes ds as an abbreviation for $\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$ for both the case of constant c ("the usual theory of relativity") and the spatially variable but static $c(x,y,z)$.

The significance of this notation does not emerge until Section 2, which treats "Equations for the motion of a particle in an arbitrary gravitational field. Characterization of the latter":

> With the introduction of a spatial variability of the quantity c we have passed beyond the framework of the theory that is now designated as "the theory of relativity"; for the expression designated by ds no longer behaves as an invariant under linear orthogonal transformations of the coordinates. ...
>
> If we introduce a new spacetime system $K'(x', y', z', t')$ by means of an arbitrary substitution.
>
> $$x' = x'(x, y, z, t)$$
> $$y' = y'(x, y, z, t)$$
> $$z' = z'(x, y, z, t)$$
> $$t' = t' = (x, y, z, t)$$
>
> and if the gravitational field in the original system K is static, then under this substitution equation (1) goes over into an equation of the form
>
> $$\delta\left\{\int ds'\right\} = 0,$$
>
> where
>
> $$ds'^2 = g_{11}dx'^2 + g_{22}dy'^2 + \ldots + 2g_{12}dx'dy' + \ldots$$
>
> and the quantities $g_{\mu\nu}$ are functions of x', y', z', t' ...
>
> Thus we arrive at the interpretation that, in the general case, the gravitational field is characterized by ten spacetime functions ... . (Einstein and Grossmann 1913, 6)

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75 "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation I. Physikalischer Teil von Albert Einstein," (Einstein and Grossmann 1913); reprinted in (CPAE 4, 303–323). For a discussion of this paper, see the Editorial Note "Einstein on Gravitation and Relativity: The Collaboration with Marcel Grossmann," (CPAE 4, 294–301). See also the Editorial Note "Einstein's Research Notes on a Generalized Theory of Relativity,"(CPAE 4, 192–199, especially Section II, 193–195).

Einstein proceeds to derive the equations of motion from the generalized Hamiltonian function (which we would now call the Lagrangian):

$$H = -m\frac{ds}{dt} = -m\sqrt{g_{11}\dot{x}_1^2 + \ldots + \ldots + 2g_{12}\dot{x}_1\dot{x}_2 + \ldots + \ldots + 2g_{14}\dot{x}_4 + \ldots + g_{44}}$$

[where $\dot{x}_1 = \frac{dx_1}{dt}$, etc. -JS].

He writes the three Lagrange equations for the three spatial coordinates x_i, derives from them expressions for the momentum of the particle and the force exerted on it by the gravitational field; and then derives the energy of the particle by performing the usual Legendre transformation on H. He closes this discussion by noting that:

> In the usual relativity theory only linear orthogonal substitutions are permitted. It will be shown that we are able to set up equations for the influence of the gravitational field on material processes that behave covariantly under arbitrary substitutions. (Einstein and Grossmann 1913, 7)

Up to this point, there is nothing in Einstein's results that depends on the interpretation of the $g_{\mu\nu}$ as anything more than a generalization of his 1912 results for the static gravitational field to their form in an arbitrary reference frame. (Remember that, for Einstein, a change of coordinates amounts to a change of spacetime reference frame.) In such a frame, the single static gravitational potential c is transformed into ten functions $g_{\mu\nu}$.

It is only in the next section of the paper, "Significance of the Fundamental Tensor $g_{\mu\nu}$ For the Measurement of Space and Time," that he proceeds to the geometrical interpretation of these functions in spacetime. After recalling that the time coordinate had already lost its immediate physical significance in a static gravitational field, he continued:

> In this connection, we remark that ds is to be understood as an invariant measure for the interval [*Abstand*] between two neighboring spacetime points. (Einstein and Grossmann 1913, 8)

Presumably, this is what his results in the "Supplement" suggested to him almost immediately: if the integrand is interpreted as the interval ds between neighboring points, then the variational principle can be interpreted as giving rise to the equation for a geodesic in the resulting non-flat space time. In a later reminiscence (Einstein 1955),[76] Einstein stated:

> The equivalence principle allows us ... to introduce non-linear coordinate transformations in such a [four-dimensional] space [with (pseudo)-Euclidean metric]; that is, non-Cartesian ("curvilinear") coordinates. The pseudo-Euclidean metric then takes the general form:
>
> $$ds^2 = \sum g_{ik}dx_i dx_k$$

76 Cited from those given in (Seelig 1955).

> summed over the indices i and k (from 1–4). These g_{ik} are then functions of the four coordinates, which according to the equivalence principle describe not only the metric but also the "gravitational field." ... This formulation so far applies only to the case of pseudo-Euclidean space. It indicates clearly, however, how to attain the transition to gravitational fields of a more general type. Here too the gravitational field is to be described by a type of metric, that is a symmetric tensor field g_{ik} ... The problem of gravitation was thereby reduced to a purely mathematical one. Do differential equations exist for the g_{ik} that are invariant under non-linear coordinate transformations? Such differential equations and *only* such could be considered as field equations for the gravitational field. The equation of motion of a particle was then given by the equation for a geodesic line.
>
> With this task in mind, I turned to my old student friend Marcel Großmann, who had in the meantime become Professor of Mathematics at the ETH (pp. 14–15).

Einstein here makes a rather precise claim about what he had accomplished before turning to Grossmann upon his move back to Zurich at the end of July 1912. Earlier, in 1923, he had made a similar claim, but with a significant addition—a reference to Gauss:

> I first had the decisive idea of the analogy between the mathematical problems connected with the theory and the Gaussian theory of surfaces in 1912 after my return to Zurich, initially without knowing Riemann's and Ricci's or Levi Civita's investigations.[77]

In the 1955 reminiscence, Einstein noted that Carl Friedrich Geiser's course on differential geometry at the ETH played an important role in his thinking; it was in that course that he learned about Gauss' theory of surfaces, based on analysis of the distance ds between neighboring points on a surface, expressed in terms of an arbitrary coordinate system on the surface, thereafter often called Gaussian coordinates.[78]

The spacetime interval ds Einstein introduced represents a generalization of what was often referred to in differential geometry as the "line element." The term "element" appears to go back to Monge, who speaks of the "elements" [*élémens*] of a curve in space. Coolidge comments: "An élémen is an infinitesimally short chord."[79] Gauss[80] makes the concept of what he calls a "line element" [he uses both "*Linienelement*" and "*Linearelement*"—see (Gauss 1881, 341–347)] connecting a pair of points on a two-dimensional surface central to his theory of surfaces, and

77 In the Preface to the Czech edition of his popular book on relativity. The German text is in (CPAE 6, 535, n. [4]).

78 In (Einstein 1955), Einstein described the lectures as "true masterpieces of pedagogical art, which later helped me very much when wrestling with general relativity" (pp. 10–11). In a letter of 24 April 1930 to Walter Leich, Einstein wrote: "Geiser was dry only in the large lectures, otherwise I owe him the most of all." For an outline of the contents of Geiser's course on "Infinitesimalgeometrie" given in the Winter Semester of 1897/1898 and the Summer Semester of 1898, based on Grossmann's lecture notes, see (CPAE 1, 365–366). Grossmann's notes are preserved in the ETH Bibliothek, Hs 421: 15 & 16.

79 See (Coolidge 1940), "Book III Differential Geometry," Chapters I, II and III, 318–387. The citations are from p. 322.

shows that it may be used to define the intrinsic properties of the surface, such as its curvature. Bianchi-Lukat (Bianchi 1910, 60–61)[81] defines ds as the "element of a curve" [*Bogenelement*], and comments:

> Since the expression for ds given by [$ds^2 = Edu^2 + 2Fdudv + Gdv^2$, the right hand side being the square of Gauss' expression for the line element] holds for any arbitrary curve on the surface, it is designated the *line element* of the surface. Gauss' ideas were generalized to n- dimensional manifolds by Riemann. (Riemann 1868)[82]

The basic ideas of Gauss' theory of surfaces are reproduced in Grossmann's notes on Geiser's course on differential geometry [*Infinitesimalgeometrie*].[83] Reich indicates some of the high points:

> Geiser treats the line element and its special form in different coordinate systems especially intensively,

indicating that he used the notation ds^2 for it.

> The curvature of surfaces and especially the Gaussian measure of curvature, which Geiser derives in Gauss' fashion and with Gauss' notation, is a particular theme of the semester. The result reads: 'if
>
> $$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$
>
> then the measure of curvature depends solely on E, F, G and their derivatives.' ... A further important point are geodesic lines. After a longish introduction, Geiser goes into the 'differential equation for geodesic lines,' ... It appears important to me that Geiser offers not only the geometrical aspect but also argues invariant-theoretically, as in the case of the metric, the measure of curvature and of geodesic lines. (Reich 1994, 164–165)

Geiser included a derivation of the equation for a geodesic on a surface by variation of the integral $\int ds$, where

$$ds^2 = \sqrt{E\,dp^2 + 2F\,dp\,dq + G\,dq^2},$$

along some curve $q = \psi(p)$ to find its minimum.[84] Comparison of this with Einstein's variational principle for the equations of motion of a particle in a static gravitational field could have suggested the analogy between Gauss' theory of surfaces and Einstein's theory of the static gravitational field.

80 "Disquisitiones generales circa superficies curvas," (Gauss 1828); reprinted in (Gauss 1881, 217–258). A valuable notice ["*Anzeige*"] by Gauss appeared in (Gauss 1827); reprinted in (Gauss 1881, 341–347). The basic idea of using the line element to investigate the properties of a surface had already been used in Gauss' 1822 Prize Essay, (Gauss 1825); reprinted in (Gauss 1881, 189–216). For discussions, see (Coolidge 1940, ch. III, sec. 1, 355–359) and (Stäckel 1918, Heft V, 25–142, sec. V, "Die allgemeine Lehre von den krummen Flächen," 104–138).

81 As we shall see below, this work was consulted by Marcel Grossmann.

82 This refers to the posthumous publication of Riemann's *Habilitationsschrift* of 1854.

83 These notes are described in (Reich 1994, 163–166).

84 Grossmann notes for 10 June 1898.

So much for the mathematical literature. In the physics literature, the equivalent proper time element $d\tau$ had been introduced by Minkowski for timelike worldlines.[85] But neither the concept of, nor the notation for, the line element ds or the proper time element $d\tau$ occurs in the works of Sommerfeld or Laue developing special-relativistic vector and tensor analysis (which, as we have seen Einstein studied) until after 1912.[86] Nor does it occur in Einstein's summary of vector and tensor analysis in his unpublished review of the special theory (discussed earlier).[87]

However, the concept and even the term, were beginning to appear in the physics literature in connection with the discussion of rigid bodies.[88] Herglotz seems to have introduced them (Herglotz 1910, 394). After pointing out that Minkowski had introduced the idea of representing the spatial and temporal coordinates "as the four coordinates of a point in a fourfold-extended manifold $R_4(x, y, z, t)$," he goes on:

> Similarly a measure relation [*Maßbestimmung*] is introduced in this R_4, according to which (the velocity of light being set equal to 1) the square of the distance of two infinitely neighboring points is:
>
> $$ds^2 = dx^2 + dy^2 + dz^2 - dt^2.$$
>
> Line elements of real length $(ds^2 > 0)$ are called spatial, however those of purely imaginary length $(ds^2 < 0)$ are called timelike.

Born's next paper on the definition of a rigid body (Born 1910, 233) speaks of "a four-dimensional space $x\ y\ z\ t$ ... in which a measure relation [*Maßbestimmung*] with the line element $dx^2 + dy^2 + dz^2 - c^2dt^2$ is introduced" (p. 233).

As noted above, Einstein refers to these papers in a 1910 letter, so we may assume that he was familiar with them.[89] And, it was in response to a criticism by Einstein that Abraham wrote:[90]

> On lines 16, 17 of my note "On the Theory of Gravitation," an oversight is to be corrected, of which I became aware through a friendly communication of Mr. A. Einstein. One should read "let us consider dx, dy, dz and $du = i\ dl = i\ cdt$ as components of a displacement ds in four-dimensional space." Thus,

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85 See Section III, p. 108 of (Minkowski 1909a); it was soon reprinted as a separate booklet, (Minkowski 1909b); and then in Minkowski's *Gesammelte Abhandlungen*, vol. 2, (Minkowski 1911, 431–434).

86 Laue only introduces the proper time and uses it to define the four-velocity in the second edition of *Das Relativitätsprinzip* (Laue 1913, 57 and 69).

87 Einstein does define the four-velocity vector by the following equation:

$$G_\mu = \left\{\frac{dx_\mu}{\sqrt{-\sum dx_\sigma^2}}\right\},$$

but without any explanation of the expression in the denominator, which does not occur anywhere else in the paper (CPAE 4, 84).

88 See (Maltese and Orlando 1995).

89 "The latest relativity-theoretical investigations of Born and of Herglotz interest me very much ..." Albert Einstein to Jakob Laub, 16 March 1910, (CPAE 5, 231–233; citation from 232).

90 "Berichtigung," (Abraham 1912c). This correction was submitted to issue no. 4, which had a closing date of 2 February 1912, but was published on 15 February 1912.

$$ds^2 = dx^2 + dy^2 + dz - c^2dt^2$$

> is the square of the four-dimensional line element, in which the velocity of light c is determined by equation (6) [Abraham's relation between the gravitational potential and the speed of light–JS].[91]

Indeed, as noted above, in Paper I on the static field Einstein had written that, in using a light clock:

> we operate with a sort of local time, which Abraham designates with l. This stands in the relation
>
> $$dl = c\ dt$$
>
> to the universal time.

This is the earliest indication (the end of February 1912) that Einstein realized the need to use differentials of the two quantities in order to relate a coordinate time t to a physically measured time l (in this case, the proper time between two events at the same spatial point).

To summarize: on the basis of the mathematical and physical resources at his command, at some point in mid-1912, after generalizing the single gravitational potential c to the array of ten gravitational potentials g_{ik}, Einstein realized that they formed the coefficients of a quadratic form $\sum g_{ik}dx_i dx_k$, which could be regarded as the square of the invariant line element $(ds^2 = \sum g_{ik}dx_i dx_k)$ of a four-dimensional spacetime manifold; and that the interval ds represents a physically measurable quantity—the proper time if the interval between two events were time-like, the proper length if it were space-like (of course it would vanish for null intervals).

I suggest that it was at this point that he turned to Grossman. Continuing the quotation from the 1955 reminiscence:

> I was made aware of these [works by Ricci and Levi-Civita—JS] by my friend Großmann in Zurich, when I put to him the problem to investigate generally covariant tensors, whose components depend only on the derivatives of the coefficients of the quadratic fundamental invariant.
>
> He at once caught fire, although as a mathematician he had a somewhat skeptical stance towards physics. ... He went through the literature and soon discovered that the indicated mathematical problem had already been solved, in particular by Riemann, Ricci and Levi-Civita. This entire development was connected to the Gaussian theory of curved surfaces, in which for the first time systematic use was made of generalized coordinates. (Seelig 1955, 15, 16)

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91 Abraham reiterated this point in his next paper "Die Erhaltung der Energie und der Materie im Schwerkraftfelde," (Abraham 1912d): "As a result of the variability of c, the Lorentz group only holds in the infinitely small, so that dx, dy, dz and $du = i\ cdt$ represent the components of an infinitely small displacement in a four-dimensional space" (p. 312). Note that, in contrast to Einstein, Abraham's c may be a function of all four spacetime coordinates.

In short: While the analogy to Gaussian surface theory had occurred to Einstein *before* he consulted Grossmann, probably including the role of the line element; the connection between this theory and the later line of development from Riemann to Ricci and Levi-Civita only became clear to Einstein *after* consulting Grossmann.

Louis Kollross, another student friend of Einstein, who was also Professor of Mathematics at the ETH during this time, adds another name that belongs between those of Riemann and of Ricci and Levi-Civita:

> [Einstein] spoke to Großmann about his troubles and said to him one day: "Großmann, you must help me, otherwise I'll go crazy!" And Marcel Großmann succeeded in showing him that the mathematical instrument that he needed had been created precisely in Zurich in the year 1869 by Christoffel in the paper "On the Transformation of Homogeneous Differential Expressions of the Second Degree," published in volume 70 of "Crelle's Journal" for pure and applied mathematics.[92]

A look at Grossmann's Part II of their joint paper, confirms Kollros's recollection:

> The mathematical tool [*Hilfsmittel*] for the development of the vector analysis of a gravitational field that is characterized by the invariance of the line element
>
> $$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu$$
>
> goes back to the fundamental paper of Christoffel on the transformation of quadratic differential forms.[93]

So it was Marcel Grossmann, who introduced Einstein to the work of Ricci and Levi-Civita after Einstein's return to Zurich in early August 1912.[94] However, in his exposition Grossmann plays down the geometrical significance of vector and tensor analysis:

> In it I have purposely left geometrical methods [*Hilfsmittel*] aside, since in my opinion they contribute little to the visualization [*Veranschaulichung*] of the concepts constructed in vector analysis (p. 325).

This distinction between tensor analysis and geometrical methods is based on the distinction Ricci and Levi-Civita make between the "fundamental quadric or form" (p. 13),which they denote by ϕ, and the line element (they never use these words), denoted by ds^2, of an n-dimensional manifold, denoted by V_n (see, e.g., pp. 128, 153). They assert that: "The methods of the absolute differential calculus depend essentially on consideration of" the fundamental form (p. 133); but the geometrical interpretation of it "as the ds^2 of a surface" (p. 162) is merely one possibility.

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92 "Erinnerungen-Souvenirs," (Kollross 1955b); reprinted as "Erinnerungen eines Kommilitionen," (Kollross 1955a). Citation from p. 27.

93 "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation II. Mathematischer Teil von Marcel Grossmann," (Einstein and Grossmann 1913, 23–38); reprinted in (CPAE 4, 324–339; citation from p. 324). The paper cited is (Christoffel 1869).

94 "Méthodes de calcul différentiel absolu et leurs applications," (Ricci and Levi-Civita 1901).This paper has been translated into English in (Hermann 1975).

Grossmann's exposition of tensor analysis is based on Chapter I, "Algorithm of the Absolute Differential Calculus" (pp. 128–144), which includes discussions of covariant differentiation and of the Riemann tensor that do not depend at all upon the geometrical interpretation of the fundamental form, but rather on the theory of algebraic and differential invariants of the fundamental form and other functions (see pp. 127 of the "Preface" and Section 1 of the first chapter, "Point transformations and systems of functions," pp. 128–130).[95] This is entirely in the spirit of Christoffel's exposition of the differential invariants of a quadratic differential form in n independent variables. Only in the last paragraph of his paper does he mention "a posthumous paper of Riemann" on "the square of the line element in a space of three dimensions" (Christoffel 1869, 70). And indeed, until Levi-Civita developed the concept of parallel displacement in a manifold with metric, geometrical methods did not contribute much to the interpretation of the covariant derivative and the Riemann tensor.[96] Only starting with Chapter II of Ricci and Levi-Civita, on "Intrinsic geometry as a calculational tool," are geometrical applications to n-dimensional manifolds introduced.

Bianchi-Lukat (Bianchi 1910), another source that Grossmann mentions,[97] also separates the invariant-theoretical treatment of "Binary Quadratic Forms" in Chapter II from the geometrical treatment of "Curvilinear Coordinates on Surfaces" in Chapter III, which includes the introduction of "The Line Element of Surfaces" in Section 33.

—But I have already begun to encroach on the opening scene of Act III. If, so far, Einstein's intuition led him almost without mis-step along the highway in the search for dynamical field equations governing the behavior of the metric tensor field, the last act will depict our hero's wanderings along many a curious by-way, before regaining the high road.

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95 Only in Section 4 of this chapter, "Applications to vector analysis," is "the ds^2 of [Euclidean three-] space as the fundamental form" introduced (p. 135). The paper has a number of lapses: for example, the fundamental form is introduced on p. 130 without the name, and the notation ϕ for it is used on p. 132, before both are defined on p. 133. Most serious, the Christoffel symbols of the second kind are introduced and used to define the covariant derivative on p. 138, without ever being defined or related to the symbols of the first kind, which are defined and then used to define the covariant form of the Riemann tensor (called "the covariant system of Riemann) on p. 142.

96 For a discussion of this question, see "The Story of Newstein or: Is Gravity just Another Pretty Force?" (in vol. 4 of this series).

97 See ""Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation II. Mathematischer Teil von Marcel Grossmann," (CPAE 4, 330).

REFERENCES

Abraham, Max. 1912a. "Zur Theorie der Gravitation." *Physikalische Zeitschrift* 13: 1–4. (English translation in vol. 3 of this series.)

———. 1912b. "Das Elementargesetz der Gravitation." *Physikalische Zeitschrift* 13: 4–5.

———. 1912c. "Berichtigung (zu: Zur Theorie der Gravitation, 1912)." *Physikalische Zeitschrift* 13: 176.

———. 1912d. "Die Erhaltung der Energie und der Materie im Schwerkraftfelde." *Physikalische Zeitschrift* 13: 311–314.

———. 1912e. "Relativität und Gravitation. Erwiderung auf eine Bemerkung des Hrn A. Einstein." *Annalen der Physik* 38: 1056–1058.

Bianchi, Luigi. 1910. *Vorlesungen über Differentialgeometrie,* 2 ed. translated by M. Lukat. Leipzig/Berlin: Teubner.

Born, Max. 1909a. "Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips." *Annalen der Physik* (30) 11: 1–56.

———. 1909b. "Über die Dynamik des Elektrons in der Kinematik des Relativitätsprinzips." *Physikalische Zeitschrift* (10) 22: 814–817.

———. 1910. "Über die Definition des starren Körpers in der Kinematik des Relativitätsprinzips." *Physikalische Zeitschrift* (11) 6: 233–234.

Christoffel, Elwin Bruno. 1869. "Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades." *Journal für die reine und angewandte Mathematik* 70: 46–70.

Coolidge, Julian Lowell. 1940. *A History of Geometrical Methods*. Oxford: Clarendon Press.

CPAE 1: John Stachel, David C. Cassidy, Robert Schulmann, and Jürgen Renn (eds.), *The Collected Papers of Albert Einstein*. Vol. 1. *The Early Years, 1879–1902*. Princeton: Princeton University Press, 1987.

CPAE 2: John Stachel, David C. Cassidy, Jürgen Renn, and Robert Schulmann (eds.), *The Collected Papers of Albert Einstein*. Vol. 2. *The Swiss Years: Writings, 1900–1909*. Princeton: Princeton University Press, 1989.

CPAE 3: Martin J. Klein, A. J. Kox, Jürgen Renn, and Robert Schulmann (eds.), *The Collected Papers of Albert Einstein*. Vol. 3. *The Swiss Years: Writings, 1909–1911*. Princeton: Princeton University Press, 1993.

CPAE 4: Martin J. Klein, A. J. Kox, Jürgen Renn, and Robert Schulmann (eds.), *The Collected Papers of Albert Einstein*. Vol. 4. *The Swiss Years: Writings, 1912–1914*. Princeton: Princeton University Press, 1995.

CPAE 5: Martin J. Klein, A. J. Kox, and Robert Schulmann (eds.), *The Collected Papers of Albert Einstein*. Vol. 5. *The Swiss Years: Correspondence, 1902–1914*. Princeton: Princeton University Press, 1993.

CPAE 6: A. J. Kox, Martin J. Klein, and Robert Schulmann (eds.), *The Collected Papers of Albert Einstein*. Vol. 6. *The Berlin Years: Writings, 1914–1917*. Princeton: Princeton University Press, 1996.

CPAE 7: Michel Janssen, Robert Schulmann, József Illy, Christoph Lehner, and Diana Kormos Buchwald (eds.), *The Collected Papers of Albert Einstein*. Vol. 7. *The Berlin Years: Writings, 1918–1921*. Princeton: Princeton University Press, 2002.

CPAE 8: Robert Schulmann, A. J. Kox, Michel Janssen, and József Illy (eds.), *The Collected Papers of Albert Einstein*. Vol. 8. *The Berlin Years: Correspondence, 1914–1918*. Princeton: Princeton University Press, 1998. (Part A: 1914–1917, pp. 1–590. Part B: 1918, pp. 591–1118.)

Ehrenfest, Paul. 1909. "Gleichförmige Rotation starrer Körper und Relativitätstheorie." *Physikalische Zeitschrift* 10: 918.

———. 1913a. "On Einstein's Theory of the Stationary Gravitation Field." *Proceedings Amsterdam Academy* 15: 1187–1191.

———. 1913b. "Over Einstein's Theorie van het stationaire gravitatieveld." *Verslagen Akademie Amsterdam* 21: 1234–1239.

Einstein, Albert. 1905. "Zur Elektrodynamik bewegter Körper." *Annalen der Physik* 17: 891–921, (CPAE 2, Doc. 23).

———. 1907. "Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen." *Jahrbuch der Radioaktivität und Elektronik* 4: 411–462, (CPAE 2, Doc. 47).

———. 1908. "Berichtigungen." *Jahrbuch der Radioaktivität und Elektronik* 5: 98–99, (CPAE 2, Doc. 49).

———. 1911a. "Die Relativitäts-Theorie." *Naturforschende Gesellschaft in Zürich, Vierteljahresschrift* 56: 1–14, (CPAE 3, Doc.17).

———. 1911b. "Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes." *Annalen der Physik* 35: 898–908, (CPAE 3, Doc. 23).

———. 1912a. "Gibt es eine Gravitationswirkung, die der elektrodynamischen Induktionswirkung analog ist?" *Vierteljahresschrift für gerichtliche Medizin und öffentliches Sanitätswesen* 44: 37–40, (CPAE 4, Doc. 7).
———. 1912b. "Lichtgeschwindigkeit und Statik des Gravitationsfeldes." *Annalen der Physik* 38: 355–369, (CPAE 4, Doc. 3). (Cited in text as Paper I.)
———. 1912c. "Relativität und Gravitation. Erwiderung auf eine Bemerkung von M. Abraham." *Annalen der Physik* 38: 1059–1064, (CPAE 4, Doc. 8).
———. 1912d. "Zur Theorie des statischen Gravitationsfeldes." *Annalen der Physik* 38: 443–458, (CPAE 4, Doc. 4). (Cited in text as Paper II.)
———. 1916a. "Die Grundlage der allgemeinen Relativitätstheorie." *Annalen der Physik* 49: 769–822, (CPAE 6, Doc. 30).
———. 1916b. "Ernst Mach." *Physikalische Zeitschrift* 17: 101–104, (CPAE 6, Doc. 29).
———. 1920a. *Relativity, the Special and the General Theory, A Popular Exposition*. London: Methuen.
———. 1920b. *Über die spezielle und allgemeine Relativitätstheorie* (*Gemeinverständlich*), 5 ed. Braunschweig: Vieweg.
———. 1921. "A Brief Outline of the Development of the Theory of Relativity." *Nature* 106: 782–784.
———. 1934. Einiges über die Entstehung der allgemeinen Relativitätstheorie. In *Mein Weltbild*, 248–256. Amsterdam: Querido.
———. 1949. "Autobiographical Notes." In P. A. Schilpp (ed.), *Albert Einstein: Philosopher-Scientist*, 2–94. Open Court: La Salle/Cambridge University Press, London.
———. 1955. "Erinnerungen-Souvenirs." *Schweizerische Hochschulzeitung* (Sonderheft) 28: 145–153.
———. 1979. *Autobiographical Notes: A Centennial Edition*. Open Court: La Salle and Chicago.
Einstein, Albert, and Marcel Grossmann. 1913. *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation*. Leipzig: Teubner, (CPAE 4, Doc. 13).
Einstein, Albert, and Jakob Laub. 1908. "Über die elektromagnetischen Grundgleichungen für bewegte Körper." *Annalen der Physik* 26: 532–540, (CPAE 2, 509–517).
Frank, Michael L. 1912. "Bemerkung betreffs der Lichtausbreitung in Kraftfeldern." *Physikalische Zeitschrift* 13: 544–545.
Gauss, Carl Friedrich. 1825. "Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer anderen gegebenen Fläche so abzubilden dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird." *Astronomische Abhandlungen* Heft 3, H. C. Schumacher (ed.). Altona.
———. 1827. "Anzeige: Disquisitiones generales circa superficies curvas." *Göttingische gelehrte Anzeigen* Stück 177: 1761–1768.
———. 1828. "Disquisitiones generales circa superficies curvas." *Commentationes societatis regiae scientarium Gottingensis recentiores* 6: 99–146.
———. 1881. Disquisitiones generales circa superficies curvas. In *Werke*, vol. 4. Göttingen: Königliche Gesellschaft der Wissenschaften.
Herglotz, Gustav. 1910. "Über den vom Standpunkt des Relativitätsprinzips aus als "starr" zu bezeichnenden Körper." *Annalen der Physik* 31: 393–415.
Hermann, Robert. 1975. "Ricci and Levi-Civita's Tensor Analysis Paper," translation and comments by R. Hermann. In *Lie Groups: History, Frontiers and Applications*. Series A, vol. 2. Brookline: Math Sci Press.
Howard, Don, and John Stachel (eds.). 1989. *Einstein and the History of General Relativity.* (*Einstein Studies*, Vol. 1) Boston: Birkhäuser.
Kaluza, Theodor. 1910. "Zur Relativitätstheorie." *Physikalische Zeitschrift* (11) 21/22: 977–978.
Kollross, Louis. 1955a. "Erinnerungen eines Kommilitionen." In (Seelig 1955).
———. 1955b. "Erinnerungen-Souvenirs." *Schweizerische Hochschulzeitung* (Sonderheft) 28: 169–173.
Laue, Max von. 1911. *Das Relativitätsprinzip*. Braunschweig: Vieweg.
———. 1913. *Das Relativitätsprinzip*, 2. ed. Braunschweig: Vieweg.
Lorentz, Hendrik Antoon. 1899–1900a. "Beschouwingen over de zwaartekracht." *Verslagen van de Gewone Vergaderingen der Wis- en Natuurkundige Afdeeling, Koninklijke Akademie van Wetenschappen te Amsterdam* 8: 603–620.
———. 1899–1900b. "Considerations on Gravitation." *Proceedings of the Section of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam* 2: 559–574.
Maltese, Giulio, and Lucia Orlando. 1995. "The Definition of Rigidity in the Special Theory of Relativity and the Genesis of the General Theory of Relativity." *Studies in the History and the Philosophy of Modern Physics* 26B: 263–303.
Minkowski, Hermann. 1908. "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern." *Nachrichten der Königliche Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse* 53–111. (English translation of excerpts in vol. 3 of this series.)
———. 1909a. "Raum und Zeit." *Physikalische Zeitschrift* (10) 3: 104–111.

———. 1909b. *Raum und Zeit*. Leipzig: Teubner.
———. 1911. *Gesammelte Abhandlungen*, D. Hilbert (ed.), vol. 2. Leipzig: Teubner.
Norton, John. 1985. "What was Einstein's Principle of Equivalence?" *Studies in the History and Philosophy of Science* 16: 203–246.
———. 1992. "Einstein, Nordström and the Early Demise of Scalar, Lorentz Covariant Theories of Gravitation." *Archives for History of Exact Sciences* 45: 17–94.
Pais, Abraham. 1982. *'Subtle is the Lord ...': the Science and the Life of Albert Einstein*. Oxford: Clarendon Press /New York: Oxford University Press.
Reich, Karin. 1994. *Die Entwicklung des Tensorkalküls: vom absoluten Differentialkalkül zur Relativitätstheorie*. *Science Networks: Historical Studies*, vol. 11. Basel/Boston/Berlin: Birkhäuser.
Ricci, Gregorio, and Tullio Levi-Civita. 1901. "Méthodes de calcul différentiel absolu et leurs applications." *Mathematische Annalen* 54: 125–201.
Riemann, Bernhard. 1868. "Ueber die Hypothesen, welche der Geometrie zugrunde liegen." *Abhandlungen der königlichen Gesellschaft der Wissenschaften zu Göttingen* 13: 133–150.
Seelig, Carl (ed.). 1955. *Helle Zeiten - Dunkle Zeiten: In Memoriam Albert Einstein*. Zürich/Stuttgart/Wien: Europa Verlag.
———. 1981. *Mein Weltbild*. Frankfurt am Main/Berlin/Wien: Ullstein.
Sommerfeld, Arnold. 1910a. "Zur Relativitätstheorie I. Vierdimensionale Vektoralgebra." *Annalen der Physik* 32: 749–776.
———. 1910b. "Zur Relativitätstheorie II. Vierdimensionale Vektoranalysis." *Annalen der Physik* 33: 649–689.
Stachel, John. 1980. "Einstein and the Rigidly Rotating Disk." In A. Held (ed.), *General Relativity and Gravitation One Hundred Years After the Birth of Albert Einstein*, vol. 1, 1–15. New York/London: Plenum Press.
———. 1982. "Einstein." *Science* 218: 989–990.
———. 1989. "Einstein's Search for General Covariance, 1912–1915." In D. Howard and J. Stachel (eds.), *Einstein and the History of General Relativity: based on the Proceedings of the 1986 Osgood Hill Conference, North Andover, Massachusetts, 8–11 May 1986*, 63–100. (*Einstein Studies,* Vol. 1) Boston: Birkhäuser.
———. 2002. *Einstein from 'B' to 'Z'*. (*Einstein Studies,* Vol. 9). Boston: Birkhäuser.
Stäckel, Paul. 1918. "Gauß als Geometer." In F. Klein et. al. (eds.), *Materialien für eine wissenschaftliche Biographie von Gauß*. Leibzig: Teubner.
Walter, Scott. 1999. "The Non-Euclidean Style of Minkowskian Relativity." In J. Gray (ed.), *The Symbolic Universe: Geometry and Physics 1890–1930*, 91–127. Oxford: Oxford University Press.