A Shape Dynamics Tutorial

(v1.0 work in progress)

Flavio Mercati

Perimeter Institute for Theoretical Physics,

31 Caroline Street North, Waterloo, ON, N2L 2Y5 Canada.

email: fmercati@perimeterinstitute.ca

Contents

Abstract

Shape Dynamics (SD) is a new theory of gravity that is based on fewer and more fundamental first principles than General Relativity (GR). The most important feature of SD is the replacement of GR's relativity of simultaneity with a more tractable gauge symmetry, namely invariance under spatial conformal transformations. This Tutorial contains both a quick introduction for readers curious about SD and a detailed walk-through of the historical and conceptual motivations for the theory, its logical development from first principles and an in-depth description of its present status. The Tutorial is sufficiently self-contained for an undergrad student with some basic background in General Relativity and Lagrangian/Hamiltonian mechanics. It is intended both as a reference text for students approaching the subject, and as a review article for researchers interested in the theory.

1 Introduction

1.1 Foreword

The main part of this introduction is in Sec. 1.3 (Shape Dynamics in a nutshell), which I have tried to make into a no-nonsense quick entry to the basic ideas of SD. It serves a dual purpose: on the one hand, students interested in Shape Dynamics will have a brief overview of what this theory is about and what we hope to do with it; on the other, researchers curious about SD will find in Sec. 1.3 a description of the theory that is short but, hopefully, complete enough for them to decide whether these ideas are worth examining in depth. The minimum of notions needed to understand the core ideas of SD will be outlined with the aim of making the section as self-contained as possible. All the concepts will be explained in detail in the rest of this Tutorial, taking an 'historico-pedagogical' perspective and introducing them at the appropriate points of the story. Sec. 1.3 includes an ultra-quick outline of basic concepts needed to understand SD, which are not part of normal undergrad curricula (like constrained Hamiltonian systems and gauge theories). However the Section is limited to just a few pages so it can be read quickly by experts, and this outline is by no means sufficient to understand properly those concepts. Its purpose is to give the undergrad readers a flavour of the background knowledge that is necessary to understand SD and get the overall drift. Everything will be fully explained in the body of the text.

Part I shows where SD comes from: we consider it as the most advanced stage of the *relational program,* which seeks to eliminate all absolute structures from physics. With absolute structures I mean anything that determines physical phenomena but is not determined by them. The chief example is Newton's absolute space and time (or, in modern terms, inertial frames of reference). The battlefield of Newton's absolutes has seen giants of science fighting the absolute-vs-relative debate: Galileo, Descartes, Newton himself, Leibniz, Mach, Poincaré and Einstein. Another example is scale, or size: Shape Dynamics tries to eliminate precisely this absolute structure from physics. One could imagine pushing this program further in the future: what determines all the physical constants?

In Sec. 2 I will explain in detail the fundamental problem of Newtonian dynamics: everything is based on the law of inertia, which in turn relies on the concepts of rest and uniform motion, but these concepts are not defined by Newton. Section 3 makes the problem with Newton's construction precise. Stating in a mathematically precise way the defect of Newton's theory was an incredibly hard problem; after more then two centuries Henri Poincaré did it. But even Poincaré's formulation (which we call the 'Mach–Poincaré principle) wasn't recognized for what it is until the work of Barbour and Bertotti, in the 1960's.

Part II deals with relational dynamics in the simpler framework of systems of point particles. Relational dynamics is a reformulation of dynamics which satisfies the Mach–Poincaré principle, as formulated by Barbour and Bertotti. It uses specific techniques that were invented on purpose, in particular that of 'best matching'. In Sec. 4 best matching is introduced at an intuitive level, while Sec. 5 details it using the language of Principal Fibre Bundles, which are introduced to the reader. Sec. 6 describes the Hamiltonian formulation of best matching, and links it to modern gauge theory. The techniques developed by Dirac for Hamiltonian constrained systems are needed in this section, and are therefore briefly explained.

Part III deals with the more advanced framework of field theory. The first Section (A) contains useful background material on the Hamiltonian formulation of General Relativity due to Arnowitt, Deser and Misner. This is helpful to make connection, in the following sections, to GR. Section 7 details (in my more modern language) a series of results due to Barbour, O'Murchadha, Foster, Anderson, and Kelleher. These results are quite striking: they show that the principles of relational field theory alone are sufficient to derive GR, the general and special relativity principles, the universality of the light cone, Maxwell's electromagnetism, the gauge principle and Yang–Mills theory. Section 8 is again background material: it presents York's method for the solution of the initial-value problem in GR. This provides an important input for the formulation of Shape Dynamics. The last Section of this Part, 9, deals with work I have done together with E. Anderson, and finally makes the connection from relational field theory to Shape Dynamics, showing that it arises uniquely from the principles of relational field theory and the Mach–Poincaré principle.

In Part IV Shape Dynamics is finally formulated in its current form. I start with a brief account of the way the ideas at the basis of SD were developed in Sec. 10, then in Sec. 11 I proceed to derive the equations of SD from the point I left the theory in Sec. 9. In Sec. 11.1 I discuss the physical degrees of freedom of SD, which are the conformally invariant properties of a 3-dimensional manifold, and their conjugate momenta. In Sec. 11.2 I explain how SD represents a simple solution to the problem of time of Quantum Gravity, and how one reconstructs the familiar 4-dimensional spacetime description of GR from a solution of SD. Finally, in Sec. 12 I describe a simplified regime of SD, namely that of asymptotically flat solutions, which can be understood as a good approximation of proper (spatially compact) solutions of SD for a short interval of time and in a small patch of space. This allows me to discuss a first physical application of SD: the description of black holes in Sec. 12.1, which turn out to be 'wormholes' connecting two universes. This is a clear difference between SD and GR, and it is particularly instructive of the inequivalence between the two theories. For the moment I will stop here with applications, but many more are upcoming and will be included in this Tutorial as soon as they reach a sufficient degree of maturity and solidity.

The next and final Part of the Tutorial contains the appendices, which are divided into a first, major Appendix, A, with a brief account of the Hamiltonian formulation of General Relativity due to Arnowitt, Deser and Misner. This is the main tool of Canonical General Relativity and is the theory we have to compare classical SD to. In this appendix I give a standard derivation of this theory starting from GR and the Einstein–Hilbert action. The same theory can be deduced from the axioms of relational field theory without presupposing spacetime and without starting from the Einstein–Hilbert action, as was done in Sec. 7. This derivation assumes less and should be considered more fundamental than that of Arnowitt, Deser and Misner, however I felt that the junior readers should be aware of the standard derivation. Finally, Appendix B contains a series of results and derivations that are useful and referenced to throughout the text, but which are moved to the end of the Tutorial for the sake of clarity of exposition.

Acknowledgements

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1.2 Notation

In the text we use a notation according to which the Greek indices $\mu, \nu, \dots$ go from 0 to 3, while the lowercase Latin indices from the middle of the alphabet $i, j, k, \ell, m \dots$ are spatial and go from 1 to 3. We assume a Lorenzian signature $(-,+,+,+)$. The lowercase Latin indices from the beginning of the alphabet a, b, c, refer to the particle number and go from 1 to N. Three-dimensional vectors will be indicated with Latin or Greek bold letters, $\mathbf{q}, \mathbf{p}, \boldsymbol{\theta}, \boldsymbol{\omega}, \dots$, while three-dimensional matrices will be uppercase Roman or Greek $\Omega, \Theta, \mathrm{U}, \mathrm{I}, \dots$. The spatial Laplacian $g_{ij}\nabla^i\nabla^j$ will be indicated with the symbol Δ, while for the d'Alembertian $g_{\mu\nu}\nabla^\mu\nabla^\nu$ I'll use the symbol $\Box$. The (spatial) conformal Laplacian $8\Delta - R$ will be indicated with the symbol $\bigcirc$. The vector Laplacian $\delta^i{}_j g_{kl}\nabla^k\nabla^l$ will be indicated with the same symbol for the scalar Laplacian Δ without indices, while the symbol for the conformal vector Laplacian $\nabla_j\nabla^i + \delta^i{}_j\Delta - \frac{2}{3}\nabla^i\nabla_j$ will be $\bigcirc^i{}_j$.

1.3 Shape Dynamics in a nutshell

Shape Dynamics (SD) is a field theory that describes gravity in a different way than General Relativity (GR). The differences between the two theories are subtle, though: in most situations they are indistinguishable.

SD is a gauge theory of spatial conformal (Weyl) symmetry

SD and GR are two different gauge theories defined in the same phase space that both admit a particular gauge fixing in which they coincide. This does not guarantee complete equivalence between the two theories: a gauge fixing is in general not compatible with every solution of a theory, in particular due to global issues. The equivalence between SD and GR therefore fails in some situations.

What distinguishes SD from GR as a fundamental theory of gravity is its different *ontology*.

First, SD does without spacetime: the existence of a pseudo-Riemannian 4-dimensional manifold with Lorentzian signature is not assumed among the axioms of the theory. Instead, the primary entities in SD are three-dimensional geometries that are fitted together by relational principles into a 'stack' whose structural properties can be identified in some but not all cases with those of a four-dimensional spacetime which satisfies Einstein's field equations. The closest agreement with GR occurs if the three-geometries are spatially closed, when the relational principles of SD are fully implemented. However, there is

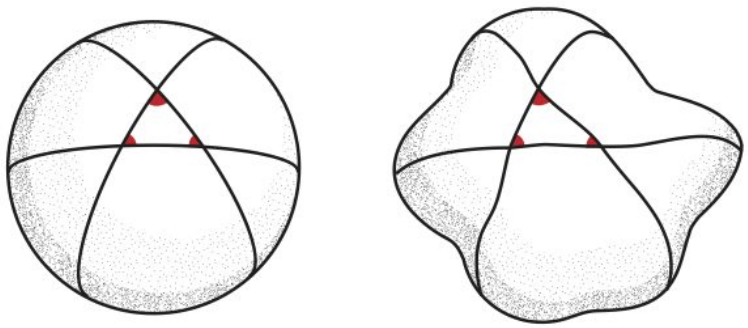

Figure 1: Conformal transformation of a 2-dimensional sphere. The triangle defined by the intersection of three curves is transformed in such a way that its area and the lengths of its three edges are changed, but the three internal angles (in red) are left invariant.

also interest in partial implementation of SD's relational principles in the case in which the three-geometries are asymptotically flat.

Second, the spatial geometries which make the configuration space of SD are not Riemannian. They are *conformal geometries,* defined as equivalence classes of metrics under position-dependent conformal transformations (sometimes called 'Weyl trasformations' ; the fourth power of ϕ is chosen to simplify the transformation law of the scalar curvature R) :

$$\{g_{ij} \sim g'_{ij} \quad \text{if} \quad g'_{ij} = \phi^4 \, g_{ij} \,, \quad \phi(x) > 0 \; \forall x\} \,. \tag{1}$$

Conformal transformations change lengths and preserve only angles (see Fig. 1). Therefore a conformal geometry presupposes less than a Riemannian geometry, for which lengths determined by the metric are considered to be physical. What is physical in SD is the *conformal structure*, which is the *angle-determining* part of the metric. Lengths can be changed arbitrarily and locally by a conformal transformation, which is a gauge transformation for SD.

So SD assumes less structure than GR, but it is in one sense a minimalistic lifting of assumptions: the next thing in order of simplicity after Riemannian geometry is conformal geometry. Some of the other approaches to quantum gravity are decidedly more radical as regards the amount of structure they assume: either much more (string theory) or much less (causal sets).

SD is based on fewer and more basic kinematical first principles than GR (see Part II):

Spatial relationalism: the positions and sizes of objects are defined relative to each other. This determines what the physical configuration space is (see Sec. 3.2).

Temporal relationalism: the flow of time is solely due to physical changes (see Sec. 5.1).

The *Mach–Poincaré principle:* a point and a direction (or tangent vector, in its weak form) in the physical configuration space are sufficient to uniquely specify the solution (see Sec. 3.4).

There is no need for general covariance, the relativity principle, the existence of spacetime, the existence of measuring rods and clocks. These concepts emerge from the solutions of SD as characteristic behaviours or useful approximations. In this sense SD is more fundamental than GR because it achieves the same with less. See Part III for the full construction of SD starting from its three first principles.

A common mistake is to regard SD just as a gauge-fixing of General Relativity. It is easy to see that this is not the case: *there are solutions of SD that are not solutions of GR, and vice-versa.* A satisfactory understanding of the GR solutions which SD excludes and of the SD solutions which GR excludes is still lacking. At this point we have just studied some examples, but the results are so far encouraging. For example the vacuum, asymptotically flat and spherically symmetric solution of SD is not Schwarzschild, it is instead a 'wormhole', which is not a vacuum solution of GR and which does not have the singularity that characterizes Schwarzschild's solution at the origin. See Sec. 12.1 for the derivation of the wormhole in SD. The equivalence between GR and SD also breaks down in some homogeneous cosmological models, which in GR possess a singularity at which spacetime ceases to exist, but in SD this singularity is a sort of coordinate artifact (it's a singularity of the gauge choice) and the solution can be continued beyond it.

Let's now have a brief look at what exactly SD looks like.

Gauge theories are constrained Hamiltonian systems

SD is more naturally formulated as a gauge theory in the Hamiltonian language. Gauge theories are theories with redundancies: one uses more degrees of freedom than necessary in order to attain a simpler and *local* description. In the Hamiltonian picture, this translates into *nonholonomic constraints:* functions of the canonical variables $\chi = \chi(p, q)$ (with some dependency on the momenta everywhere on phase space) which need to vanish on the solutions of the theory $\chi(p, q) \approx 0$.[1] A single constraint identifies a codimension-1 hypersurface

[1] With '$\approx$' we mean that the equation holds on the solutions of the equations of motion, following Dirac's notation.

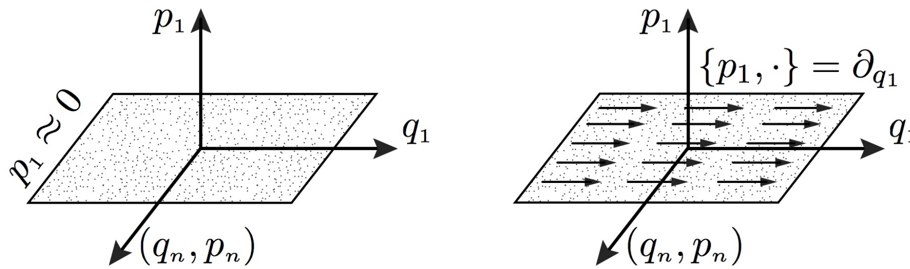

Figure 2: The constraint surface of a gauge constraint $\chi = p_1$ is represented in phase space, where I put p_1 and q_1 on two axes, and all the other phase-space variables (q_n, p_n), $n = 2, 3 \dots$ are represented collectively on the third axis. On the right, I show the vector field generated by p_1 through Poisson-brackets, $\{p_1, \cdot\} = \frac{\partial}{\partial q_1}$, which points towards the q_1 direction. The vector field is parallel to the constraint surface, and its integral curves (the gauge orbits) lie on it.

in phase space, the *constraint surface*, on which the solutions of the theory are localized. For example, if a gauge constraint can be written as $\chi = p_1$ (one of the momenta), as is always *locally* possible, the constraint surface is the hyperplane $p_1 \approx 0$ shown in Fig. 2. But p_1 also plays the role of the *generator* of gauge transformations, which happen to be the translations in the q_1 direction: through the Poisson bracket it defines a vector field on phase space $\{p_1, \cdot\} = \frac{\partial}{\partial q_1}$, which is parallel to the q_1 axis (see Fig. 2). This vector field generates infinitesimal transformations on phase space (translations in the q_1 direction), and its integral curves are the *gauge orbits* of the transformations. All the points on these curves are gauge-equivalent (they are related by gauge transformations: they have different representations but the same physical content). Moreover, the vector field $\frac{\partial}{\partial q_1}$ is parallel to the constraint surface $p_1 \approx 0$ by construction, and its integral curves lie on it. The physical meaning of a gauge constraint $\chi = p_1$ is that the q_1 coordinate is unphysical, like the non-gauge-invariant part of the electromagnetic potentials $\mathbf{A}$ and φ, or like the coordinates of the centre of mass of the whole Universe.

Since the q_1 coordinate is not physical, we can assign it any value along the solution without changing anything physical. It is often useful (and necessary in quantum mechanics) to fix the value of q_1 by some convention. The standard way of doing it is by choosing a *gauge fixing:* we specify the value of q_1 as a function of the other variables, $q_1 = q_1(q_2, p_2, \dots)$. This corresponds to intersecting the constraint surface $p_1 \approx 0$ with another surface $\xi(p, q) \approx 0$ that specifies an intersection submanifold $\{p, q \;\; s.t. \;\chi \approx 0, \xi \approx 0\}$ (see Fig. 3). The gauge fixing should specify the gauge without ambiguity: it has to form

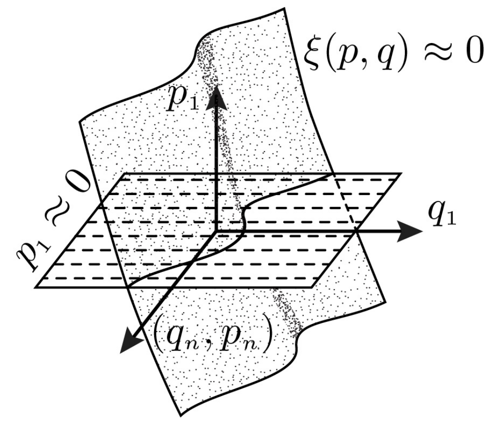

Figure 3: The concept of gauge-fixing surface: the variable q_1 is unphysical and its value can be taken arbitrarily, therefore we might choose a conventional value for q_1, to be determined by the value of all the other phase-space variables, $q_1 = q_1(q_2, p_2, \dots)$. One way to obtain this is to intersect the constraint surface $p_1 \approx 0$ with another surface, $\xi(q, p) \approx 0$, such that it is never parallel to $p_1 \approx 0$ or, at the intersection, 'runs along' the gauge orbits (represented by dashed lines on the constraint surface).

a proper intersection with $p_1 \approx 0$, and therefore cannot be parallel to it where they intersect. Moreover, at its intersection with the constraint surface $\xi \approx 0$ cannot 'run along' (be tangent to) any of the gauge orbits: in that case there would be more than one value of q_1 that corresponds to the same value of $q_2, p_2, \dots$. These two conditions define a good gauge-fixing surface. For details on constrained Hamiltonian systems and gauge theories, see Sec. 6.2.

GR as a constrained Hamiltonian theory

Arnowitt, Deser and Misner (ADM) formulated GR in the Hamiltonian language. They foliated spacetime into a stack of spatial hypersurfaces and split the 4-metric $g_{\mu\nu}$ into a spatial part g_{ij} and four additional components g_{0i} and g_{00}. The spatial metric components g_{ij} represent the canonical variables, and their momenta p^{ij} are related to the extrinsic curvature of the spatial hypersurface with respect to its embedding in spacetime. The g_{0i} and g_{00} components (or better some combinations thereof) enter the action without time derivatives, and are therefore Lagrange multipliers. They are associated with four *local* constraints (meaning one constraint per spatial point). These constraints are the so-called 'superhamiltonian' $\mathcal{H}$ and 'supermomentum' $\mathcal{H}^i$ constraint. Here I will call them the 'Hamiltonian' and the 'diffeomorphism' constraint. The diffeomorphism constraint admits a simple geometrical interpretation: its vector flow sends configuration variables into themselves (one says it generates 'point transformations' $\mathcal{H}^i : g_{ij} \to g_{ij}$), and there is no doubt about its being a gauge constraint.

For the Hamiltonian constraint things aren't that simple: it is quadratic in the momenta, and its vector flow does not admit the interpretation of a point transformation (it sends g_{ij}'s into both g_{ij}'s and p^{ij}'s). There is a large literature on the problem of interpreting $\mathcal{H}$. If it is interpreted as a gauge constraint, one would end up with the absurd conclusion that the dynamical evolution of GR is just a gauge transformation. There are also huge problems with the definition of what people call Dirac observables: quantities whose Poisson brackets with all the first-class constraints vanish on the constraint surface (meaning they must be invariant under the associated gauge transformations). In GR's case, that definition would lead to observables which are constants of motion and don't evolve ('perennials', as Kuchař called them). Kuchař advocated a different notion of observables, namely ones which are only required to be invariant under diffeomorphisms. These would evolve, but they are too many: they would depend on *three* polarizations of gravitational waves, while it is widely agreed that gravitational waves have *two* physical polarizations.

The fact that $\mathcal{H}$ is quadratic in the momenta also causes major problems in its quantization. It leads to the notorious 'Wheeler–DeWitt equation', for which there are many unsolved difficulties, above all its 'timelsss' nature, but also ordering ambiguities and coincidence limits. The ADM formulation of GR is detailed in Sec. A, and the problems with this theory which lead to the introduction of SD are explained at the end of Sec. 7 and in Sec. 8.

As illustrated in Fig. 4, SD is based on the identification of the part of $\mathcal{H}$ which is not associated with a gauge redundancy and takes it as the generator of the dynamics. The rest of $\mathcal{H}$ is interpreted as a gauge-fixing for another constraint $\mathcal{C}$. This constraint is linear in the momenta and generates genuine gauge transformations, constraining the physical degrees of freedom to be two per point.

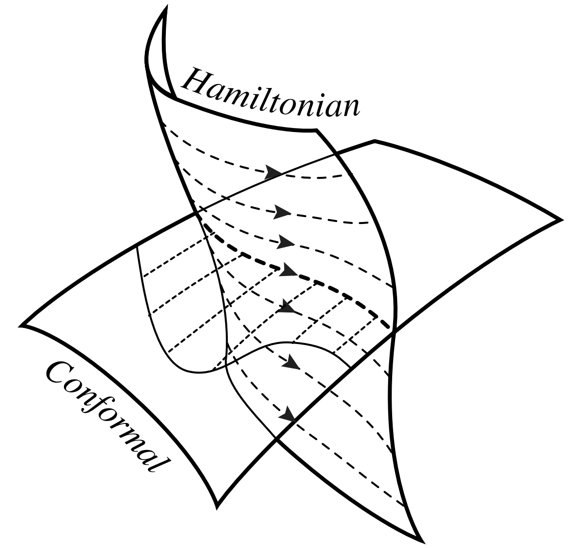

Figure 4: A schematic representation of the phase space of GR. In it, two constraints coexist, which are good gauge-fixings for each other and are both first-class with respect to the diffeomorphism constraint. One is the Hamiltonian constraint and the other is the conformal (Weyl) constraint. The Hamiltonian constraint is completely gauge-fixed by the conformal constraint except for a single residual global constraint. It Poisson-commutes with the conformal constraint and generates a vector flow on the Hamiltonian constraint surface (represented in the figure), which is parallel to the conformal constraint surface. This vector flow generates the time evolution of the system in the intersection between the two surfaces. Any solution can then be represented in an arbitrary conformal gauge by lifting it from the intersection to an arbitrary curve on the conformal constraint surface. All such lifted curves are gauge-equivalent solutions of a conformal gauge theory with conformally-invariant Hamiltonian.

Not every constraint corresponds to gauge redundancy

That this is the case is pretty obvious: think about a particle constrained on a sphere or a plane, *i.e.*, a holonomic constraint. Such a constraint obviously has nothing to do with gauge redundancy. However, there are constraints which Dirac [1, 2] argued can always be related to gauge symmetries: they are the so-called 'first-class' constraints. Being first-class means that they close an algebra under Poisson brackets with each other and with the Hamiltonian of the system. If that is the case, Dirac showed that one has freely specifiable variables in the system, one for each first-class constraint, and changing these variables does not change the solutions of the theory. But Barbour and Foster [3] have pointed out that the premises under which Dirac obtained his result do not hold in the important case in which the canonical Hamiltonian vanishes. In that case the Hamiltonian is just a linear combination of constraints, but that doesn't prevent the theory from having sensible solutions. The solutions will be curves in phase space, and will still possess one freely specifiable variable for each constraint. But one of these redundancies will not change the curve in phase space: it will just change its *parametrization*. Therefore one of the first-class constraints of the system will not be related to any gauge redundancy: there is not an associated unphysical 'q_1' direction, like in the example above. This counterexample to Dirac's statement is very important because it is realized in the theory we care about the most: General Relativity. One of the (many) constraints of GR *should not* be associated with gauge redundancy. The Barbour–Foster argument is explained at the end of Sec. 6.2.

SD reinterprets $\mathcal{H}$ as a gauge-fixing of conformal symmetry

Shape Dynamics identifies another constraint surface $\mathcal{C} \approx 0$ in the phase space of GR, which is a good gauge-fixing for the Hamiltonian constraint. This gauge-fixing, though, happens to be also a gauge symmetry generator. It generates conformal transformations (1) of the spatial metric, with the additional condition that these transformations must preserve the total volume of space $V = \int \mathrm{d}^3x\sqrt{g}$. The constraint $\mathcal{C}$, in addition, happens to close a first-class system with the diffeomorphism constraint $\mathcal{H}^i$, therefore it is a matter of opinion whether it is $\mathcal{C}$ that gauge-fixes the system $(\mathcal{H}, \mathcal{H}^i)$ or it is $\mathcal{H}$ which gauge-fixes $(\mathcal{C}, \mathcal{H}^i)$. If the real physics only lies in the intersection between $\mathcal{C} \approx 0$ and $\mathcal{H} \approx 0$ (which is the big assumption at the basis of SD, and doesn't hold if spacetime is assumed as an axiom), then the logic can be reversed and the Hamiltonian constraint can be interpreted as a special gauge-fixing for the conformal constraint. Then gravity can be reinterpreted as a gauge theory of conformal transformations, which admits a gauge-fixing that is singled out by some special properties. These properties, as I will show, have to do with the fact that it gives a 'natural' notion of scale and proper time, which agree (most of the times) with those measured by physical rods and clocks.

SD's Hamiltonian constraint

$\mathcal{H}$ and $\mathcal{C}$ do not entirely gauge-fix each other: there is a single linear combination of $\mathcal{H}(x)$ which is first-class wrt $\mathcal{C}$. This linear combination, $\mathcal{H}_{\text{global}} = \int d^3x\, N_{\text{CMC}}(x)\, \mathcal{H}(x)$, is a single global constraint whose vector flow is parallel to both the $\mathcal{C} \approx 0$ and the $\mathcal{H} \approx 0$ surfaces on their intersection. This vector flow generates an evolution in the intersection: it has to be interpreted as the generator of time evolution. It is the part of our constraints which is not associated with a gauge redundancy and is instead associated with time reparametrizations of the solutions of the theory.

The 'Linking Theory'

SD pays a price for its conceptual clarity: the generator of the evolution $\mathcal{H}_{\text{global}}$ contains the solution to a differential equation, N_{CMC}, and therefore is a nonlocal expression. But one can recover a local treatment by enlarging the phase space. SD can in fact be considered as one of the possible gauge fixings of a first-class theory which is local (its constraints are local) and lives in a larger phase space than that of GR. This phase space is obtained from that of GR by adjoining a scalar field ϕ and its conjugate momentum π. The larger theory (called 'Linking Theory') is defined by the constraints of GR, $\mathcal{H}$ and $\mathcal{H}^i$, but expressed in terms of (volume-preserving-)conformally-transformed metrics $e^{4\hat{\phi}}g_{ij}$ and momenta $e^{-4\hat{\phi}}\left[p^{ij} - \frac{1}{3}(1-e^{6\hat{\phi}})\sqrt{g}g^{ij}\int p/V\right]$, where $\hat{\phi} = \phi - \frac{1}{6}\log(\int \mathrm{d}^3x\sqrt{g}\exp(6\phi)/V)$. In addition, one has a modified conformal constraint which includes a term that transforms ϕ. The new constraint is $\mathcal{Q} = \pi - \mathcal{C}$ and generates simultaneous translations of ϕ and (volume-preserving) conformal transformations of π, so that the combination $e^{4\hat{\phi}}g_{ij}$, is left invariant. The constraint $\mathcal{Q}$ is now first-class with respect to $\mathcal{H}$ and $\mathcal{H}^i$. By completely gauge-fixing $\mathcal{Q}$, for example with the condition $\phi \approx 0$, one obtains GR. On the other hand, one can use a different gauge-fixing, namely $\pi \approx 0$, which is first-class with respect to $\mathcal{Q}$, but gauge-fixes $\mathcal{H}$ almost entirely, leaving only the global part $\mathcal{H}_{\text{global}}$ untouched.

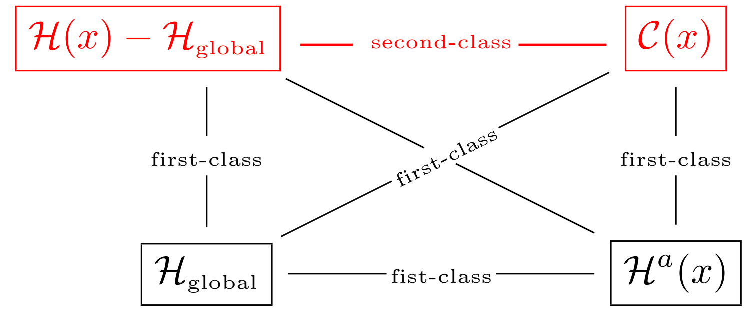

Figure 5: Scheme of the constraints of GR and of SD. GR's Hamiltonian constraint $\mathcal{H}$ has been split into the global part $\mathcal{H}_{\text{global}}$ which is first-class wrt the conformal constraint $\mathcal{C}$ and the part that is purely second class, $\mathcal{H}(x) - \mathcal{H}_{\text{global}}$. This second-class system admits two first-class subalgebras: the lower-left triangle, which constitutes the constraint algebra of GR, and the lower-right triangle, making the constraint algebra of SD.

One can then work with the Linking Theory, where all the equations of motion and constraints are local (apart from the dependence on the total volume), and work out the solutions in this framework. As long as the solution is compatible with the gauge-fixing $\pi \approx 0$, it's a legitimate SD solution.

All the details of the SD construction can be found in Sec. 9 and in part IV.

The present status of SD

Evidence of the inequivalence between GR and SD:

- Asymptotically flat, spherically symmetric vacuum solution of SD [4]: this solution is *not* Schwartzschild's spacetime, it is an wormhole connecting two asymptotically flat regions, which does not satisfy Einstein's vacuum equations on the horizon. See Section 12 for details.
- Thin-shell collapse (work in progress): the region inside the horizon in this case is compact, and it contains an expanding shell of matter which, as it expands, decompactifies the region.
- Asymptotically flat, axisymmetric, stationary vacuum solution of SD [5]: also this solution is not Kerr spacetime, it generalizes the wormhole solution with an angular momentum.
- SD in 2+1 dimensions on the torus [6]: one can evolve the solutions past a volume singularity (where $\det g_{ij} = 0$). That point marks the end of the spacetime description, but SD is capable of continuing past singularities of this kind.
- Compact, spherically symmetric solutions (work in progress).

Structural work on SD and search for a construction principle:

- Coupling SD to bosonic matter [7]: consistently done for the Standard Model fields. The uniqueness of the lapse cannot be guaranteed for any matter content: in particular, in the case of a massive scalar field there is a bound on its density above which uniqueness is broken. The physical interpretation of this is still under debate.
- Dirac observables of GR and SD coincide [8]: if SD is formulated as a Hamiltonian theory on conformal superspace with a physical Hamiltonian that is time-dependent, the problem of defining observables for Quantum Gravity is formally solved.
- SD solves the problem of time [9]: interpreting the 'York time' (which measures the expansion rate of the Universe in a particular foliation) as an 'internal' clock, SD satisfies the Mach–Poincaré principle on conformal superspace (see below). The problem of the 'frozen formalism' of Quantum Gravity is absent in this formulation. SD also provides in-principle Kuchař's perennials (work in progress).
- 'Symmetry Doubling' idea [10]: in ADM/SD phase space there are two BRST charges, associated with the symmetries of SD and GR, that gauge-fix each other. This has likely deep implications for the quantum theory.
- Simmetry doubling as a construction principle [11]: Hojman–Kuchař–Teitelboim-style derivation of SD as the unique theory with two symmetries gauge-fixing each other.
- Conformal-Cartan formulation in 2+1 dimensions [12]: first-order formulation of SD; if generalized to 3+1 dimensions, it could provide the key to the coupling of fermions to SD (work in progress).
- Path-integral quantization of SD (work in progress).

The N-body problem as the 'harmonic oscillator' of SD:

- Interesting quantum mechanics of the scale-invariant particle model [13].
- The N-body problem provides the intuition for the solution of the Problem of Time [9], the definition of in-principle Kuchař's perennials and the regularization of singularities (in the form of central collisions) (work in progress).
- Arrow of Time from friction on Shape Space [14].
- Wavefunction spontaneously 'classicalizes' and produces quantum-driven inflation [15].

Part I

Historical motivation

2 Newton's bucket

2.1 The defects of the law of inertia

Newton based his *Principia* [16] on the *law of inertia* (stated first by Galileo), which he made into the first of his three laws of motion:

> *A body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*

Assuming this law as a postulate, without first defining the notions of 'rest', 'uniform motion' and 'right (or straight) line', is inconsistent. In a Universe that is, in Barbour's words, like 'bees swarming in nothing' [17], how is one to talk about rest/uniform motion/straight lines? With respect to what?

The problem is that of establishing a notion of *equilocality*: in an ever-changing Universe, what does it mean for an object to be at the same place at different times?

Newton anticipated these criticisms in the *Scholium*[2] at the beginning of the *Principia*. He claims that rest/uniform motion/straight lines have to be defined with respect to absolute space and time:

> *I. Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.*

> *II. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is commonly taken for immovable space; [...]*

[2] A 'scholium' is an explanatory commentary.

These definitions make the *Principia* a logically consistent system, which however relies on the scientifically problematic concepts of absolute space and time. These affect the motion of material bodies in a spectacular way - through the law of inertia - but aren't affected by them. Despite these shaky grounds, Newton's Dynamics proved immensely successful over more than two centuries, and this tended to hide its foundational problems.

2.2 Leibniz's relationalism

The chief advocate for an alternative in Newton's time was Leibniz. In a correspondence with Clarke [18] (writing basically on behalf of Newton) he advocated a *relational understanding of space*, in which only the observable *relative distances* between bodies play a role

> *I will show here how men come to form to themselves the notion of space. They consider that many things exist at once and they observe in them a certain order of co-existence [...] This order is their* situation *or distance.*

We can say that Leibniz lost the argument with Clarke, mainly because he failed to provide a concrete, viable way of implementing a relational mechanics. We'll see that he had no hope of doing that, because much more sophisticated mathematics is needed than was available at the time.

Leibniz's main argument against absolute space and time – that they are not observable – was actually anticipated and countered by Newton in the *Scholium*. He claimed one could prove the existence of absolute circular motion in his famous 'bucket experiment' described as follows:

> *If a vessel, hung by a long cord, is so often turned about that the cord is strongly twisted, then filled with water, and held at rest together with the water; after, by the sudden action of another force, it is whirled about in the contrary way, and while the cord is untwisting itself, the vessel continues for some time this motion; the surface of the water will at first be plain, as before the vessel began to move; but the vessel by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little, and ascend to the sides of the vessel, forming itself into a concave figure [...] This ascent of the water shows its endeavour to recede from the axis of its motion; and the true and absolute circular motion of the water, which is here directly contrary to the relative, discovers itself, and may be measured by this endeavour. [...] And therefore, this endeavour does not depend upon any translation of the water in respect to ambient bodies, nor can true circular motion be defined by such translation.*

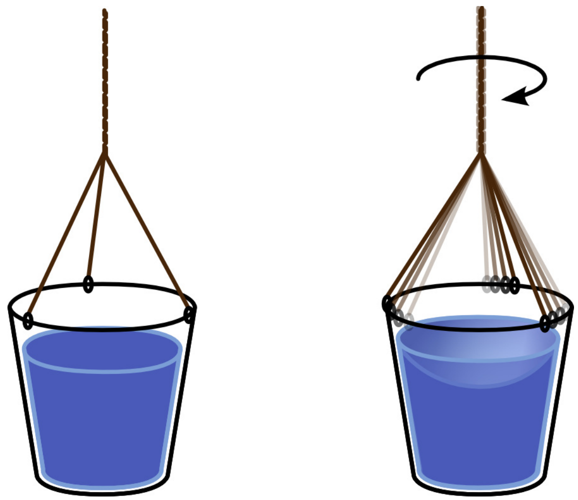

Figure 6: Newton's bucket experiment: on the left, both the bucket and the water are at rest with respect to the room. On the right, they are both rotating, and they have no motion relative to each other. Then to explain the concave shape of the water in the picture on the right, one has to invoke something external to the relative motions of water and bucket.

In this passage Newton was, besides making a serious argument, covertly attacking the philosophy of Descartes [19], who had died in 1650 but whose ideas were still widely accepted. Descartes had declared position and motion to be relative and in particular had said that the 'one true philosophical position' of a given body is defined by the matter immediately next to it. In the bucket experiment, it is obvious that the relative state of the water and the sides of the bucket is *not* the cause of the behaviour of the water: both at the beginning and at the end they are at rest wrt each other, but in one case the surface of the water is flat and in the other it's curved. Therefore the curvature of the water must be caused by something else, which Newton identifies with the circular motion wrt absolute space. This argument largely settled the issue in the mind of most scientists until the end of the XIX century.

2.3 The *Scholium* problem

Newton was aware of the difficulties inherent in his tying the first law of motion to unobservable entities like absolute space and time. Towards the end of the *Scholium* he comments

> *It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of particular bodies from the apparent; because the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses.*

But he believed that experiments like that of the bucket provided a handle on this problem:

> *Yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions.*

Newton then concludes the *Scholium* with the grand words

> *But how we are to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large in the following treatise. For to this end it was that I composed it.*

Thus, he considers that deducing the motions in absolute space from the observable relative motions to be *the fundamental problem of Dynamics*, and claims that he composed the *Principia* precisely to provide a solution to it. Remarkably, he never mentions this *Scholium problem* again in the *Principia* and certainly doesn't solve it! What is more, nobody else attempted to for very nearly 200 years.

Further reading: Newton's "Principia" [16], The Leibniz–Clarke correspondence [18], Barbour's "The Discovery of Dynamics" [19].

3 Origins of the Mach–Poincaré principle

3.1 Tait's partial solution of the Scholium problem

In 1884, Tait [20] provided a solution to the Scholium problem in the simplest case of non-interacting, i.e., inertial, point masses. I give here my personal account of it.

Say we have N point masses that don't exert any force upon each other (like a perfect gas), and we are given only the succession of the relative distances[3] between those particles r_{ab}, $a, b = 1, \ldots, N$, taken at some unspecified intervals of time. Those are $N(N-1)/2$ numbers, but they aren't all independent of each other. Newton would say that the absolute space in which they move is three-dimensional, and this constrains the r_{ab}'s to satisfy certain relationships. The simplest one is the triangle inequality between triplets: $r_{13} \leq r_{12} + r_{23}$. Then, if $N \geq 5$, there are true equalities they have to satisfy, which reduce their independent values to just $3N - 6$.[4] To convince yourselves about that just consider that if the particles were represented as points in $\mathbb{R}^3$ they would have $3N$ coordinates, but two configurations that are related by a rigid translation of the whole system (3 degrees of freedom), or a rigid rotation (3 further d.o.f.) would be equivalent, because they would give the same r_{ab}'s. So we're down to $3N - 6$ degrees of freedom.[5] This is huge data compression, from something quadratic in N to something linear. M. Lostaglio [21] convinced me that this data compression should be taken as an *experimental fact*, to which our senses are so used that it has become an intuitive truth. Geometry, in this case three-dimensional Euclidean geometry, is a synthesis of all those relationships between observables.

To come back to the *Scholium* problem, we have to determine the unobservable positions in absolute space of the N particles. These will be N vectors $\mathbf{r}_a \in \mathbb{R}^3$ ($3N$ numbers), which must be determined from the $3N-6$ independent observables that can be extracted from r_{ab} given as 'snapshots' obtained at certain unspecified times. Following Tait, we assume for the moment an external scale is given. Tait's solution exploits the assumption that the particles are not interacting, and therefore according to Newton's first law they will move uniformly in a straight line in absolute space. In a more modern language one is looking for the determination of an *inertial frame of reference*,[6] in which the first law holds. By Galilean relativity, there will be infinitely many such inertial frames of reference all related to each other by Galilean transformations. Tait's algorithm exploits this freedom and is as follows:

1. Fix the origin at the position of particle 1: then $\mathbf{r}_1 = (0, 0, 0)$.

2. Fix the origin of time $t = 0$ at the instant when particle 2 is closest to particle 1. At that instant, call $r_{12} = a$.

3. Orient the axes so that y is parallel to the worldline of particle 2, and the unit of time is given by the motion of particle 2, so that $\mathbf{r}_2 = (a, t, 0)$ (using the inertial motion of a particle as a clock is an idea due to Neumann [23]).

4. The motions of the remaining $N - 2$ particles remain unspecified. All one knows is that they will move along straight lines uniformly with respect to the time t read by Neumann's inertial clock. Their trajectories will therefore be $\mathbf{r}_a = (x_a, y_a, z_a) + (u_a, v_a, w_a)\, t$, where (x_a, y_a, z_a) and (u_a, v_a, w_a) together with a will be $6N - 11$ unspecified variables.

The conclusion we can draw is that one needs $6N - 11$ observable data to construct an inertial frame. Each 'snapshot' we are given contains only $3N - 7$ independent data ($3N - 6$ independent relative data minus the time at which each snapshot has been taken, which is unknown). Therefore two snapshots aren't enough. They provide only $2(3N - 7) = 6N - 14$ data. We're short of three numbers in order to fix a, (x_a, y_a, z_a) and (u_a, v_a, w_a). We need a third snapshot[7] to determine the inertial frame. This is especially puzzling if one considers that N can be as large as wanted, say a billion, but one would always need just three additional quantities.[8]

The additional 3 numbers that we need to specify through a third snapshot are the direction of the rotation vector of the system (which accounts for 2 degrees of freedom) and the ratio of the relative rotation to the expansion of the complete system as captured in the two snapshots. The point is that the values of r_{ab} in the two snapshots are unaffected by a rotation of arbitrary magnitude of one snapshot relative to the other about an arbitrary direction. (Since the scale is assumed given and the centroid of the points can be determined in each snapshot, the overall expansion can be deduced. The difficulty is in the relative rotation.[9]

For what follows, the important thing that emerges from Tait's analysis is not so much that Newton's Scholium problem can be solved but the fact that

[3] One could include relativity of *scale* in the picture. If the whole universe consists only of those N point particles, there is no external ruler with which we can measure sizes, so there is absolutely no meaning in concepts like 'the size of the universe'. Then, in this case, the only truly observable things in such a universe are r_{ab}/r_{cd}, the *ratios* between r_{ab}'s.

[4] $3N - 7$ if we include relativity of size.

[5] With rescalings (1 d.o.f.) we go down by a further degree of freedom to $3N - 7$.

[6] In 1885, Lange [22], using a construction principle far more complicated than Tait's, coined the expressions 'inertial system', in which bodies left to themselves move rectilinearly, and 'inertial time scale', relative to which they also move uniformly. The two concepts were later fused into the notion of an inertial frame of reference.

[7] In fact a fourth as well if $N = 3$ or 4. With only two particles, nothing can be done. Relational dynamics requires at least three particles. The Universe certainly meets that requirement!

[8] If the scale is not given, then each snapshot carries only $3N - 8$ independent data. But the unspecified variables in this case aren't as many as before: we are free to fix the scale so that $a = 1$, and therefore we only need to find (x_a, y_a, z_a) and (u_a, v_a, w_a) which are $6N - 12$. Two snapshots then fall short of 4 data.

[9] In the scale-invariant case, we do not know the direction of the relative rotation vector, the rotation rate and the expansion rate. These are the four missing quantities. Note that the particle masses do not enter the law of inertia and can only be deduced in the presence of interactions.

two ‘snapshots’ are never enough to do it. This, and the number of extra data needed due to the factors just identified, remains true in the much more realistic case of, say, N point particles known to be interacting in accordance with Newtonian gravity.[10] Of course, the task is immensely more difficult, but in principle it is solvable.

3.2 Mach’s critique of Newton

Ernst Mach was a great experimental physicist who was also convinced one needed to know the history of science in order to make real progress.[11] Being true to that belief, in 1883 he wrote a book on the history of mechanics [24] which later proved immensely influential in the development of Einstein’s General Relativity (and more recently in that of Shape Dynamics). In this book Mach criticizes Newton’s absolute space and time.

Mach’s critique of absolute time

> *It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive through the changes of things.*

Richard Feynman quipped “Time is what happens when nothing else does.” Even if meant humourously, this does rather well reflect a deeply-rooted, fundamentally Newtonian concept of time that is still widespread today. Mach would have answered to this: “If nothing happens, how can you say that time passed?” Feynman’s words express a view that is still unconsciously shared by theoretical physicists, despite being seriously questioned by GR. According to this view, in the words of Barbour [25] “in some given interval of true time the Universe could do infinitely many different things without in any way changing that interval of time.” The Machian point of view is that this is correct only if one speaks about a *subsystem* of the Universe, like the Earth or our Solar System. In that case it is true that all the matter on Earth could do many different things without changing the interval of time. But the whole Universe has to do something in order for that interval of time to be defined. It is actually the other way around: an interval of time is defined by the amount of change that the state of the Universe undergoes.

One has to ask the following question: what do we mean when we say that one second passed? Thinking about it, it becomes pretty clear that we always refer to physical things having changed. Be it the hand of a watch that has ticked once, or the Earth rotated by 1/240th of a degree, we always mean that something has changed. The modern definition of a second is “9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom” [26]. These are a lot of oscillations. The notion of a second (or, in general, duration) is useful, but it is obviously not fundamental: one can always do without it and make direct reference to comparisons of changes in the Universe. Instead of saying a car “travels a quarter of a mile in 5.78 seconds” one can equivalently say the car “travels a quarter of a mile as the Earth rotates through 0.024°”. The same holds for any other possible measurements of duration. One can imagine representing the history of the Universe as a curve on some space, each point of which represents a unique configuration (*e.g.*, one point might represent the car on the start line and the Earth with the Sun on the Zenith above Indianapolis, IN, and another point might correspond to the car on the end line, and the Sun at an angle of 0.024° from the Zenith). The Universe passes through all the points of the curve, which contain information about everything: from the position of the car to the psychological state of our brains. Then the speed at which this curve is traversed doesn’t make any difference: in this representation, if the car covers its quarter mile track twice as fast, also the Earth would rotate and our brain states will evolve at double the speed. Nothing measurable will have changed. What counts is the sequence of states the curve passes through, not the *parametrization* of the curve. I have here anticipated the precise mathematical realization of temporal relationalism that will be advocated in this tutorial: *the history of the Universe can be represented by a curve in some configuration space, independently of its parametrization.*

We’ll see later that the relationalist approach allows us to completely dispose of Newton’s absolute time, and to describe Newtonian Dynamics as a *reparametrization-invariant theory*, where there is no notion of time at all, there is just a succession of configurations without any notion of duration. Then the requirement that the equations of motion take the particularly simple form of Newton’s second law allows us to deduce a notion of time, called *ephemeris time*, which is a sort of average over all the changes in the positions of the particles in the Universe. This is the realization of Mach’s “abstraction of time from change”.

Mach’s critique of absolute space

Mach, like Leibniz and the other advocates of relationalism, was opposed to visible effects admitting an invisible cause. This is why he disagreed with Newton’s interpretation of the bucket experiment. Being a good experimentalist, Mach’s intuition told him that the thin bucket wall couldn’t possibly be responsible

[10] In fact, one more datum is needed to determine the ratio of the kinetic to the potential energy in the system at the initial instant. Also, to determine the mass ratios of the particles, $N-1$ extra data will be needed.

[11] This is part of the reason why I’m putting so much emphasis on these historical notes

for the macroscopic concavity of the water's surface, it should admit a different cause. And here came an observation which relied on the knowledge of the centuries-old practice of astronomers: Newton's laws had not been verified relative to absolute space but to the *fixed stars*, with the rotation of the Earth as a clock ('sidereal' time). Since antiquity astronomers noticed that the fixed stars (those that, unlike the wandering planets, do not change their observed relative positions on the sky) provide a reference frame with respect to which all the motions are simpler. This practice had proved to be fruitful to such an extent that, when a discrepancy was observed, it was attributed to non-uniformity of the rotation of the Earth or failure of Newton's law of gravity as happened in the 1890s when astronomers observed an anomalous acceleration in the motion of the Moon.[12] The possibility that the fixed stars didn't identify an inertial frame of reference was never taken seriously. But this is actually the case, even if only to a microscopic degree, as we'll see in a moment.

So Mach, in *The Science of Mechanics*, claimed that the cause of the concavity of the water's surface in Newton's bucket could be due to the distant stars. This would have remained a rather bizarre claim had it not been for the incredibly suggestive insight that followed:

> *No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick.*

It is then clear that Mach had in mind a sort of interaction between distant *massive* objects, and the local inertial frames. This observation made a great impression on several people, most notably Einstein, for whom it represented a major stimulus towards the formulation of General Relativity.

3.3 Hoffman's experiment

In a 1904 book Wenzel Hoffman proposed a real experiment to test Mach's idea.[13] In the absence of buckets whose sides are "several leagues thick", he

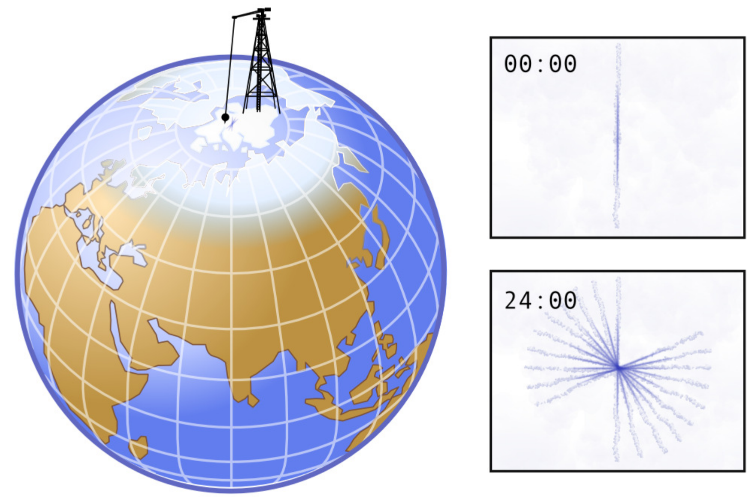

Figure 7: Hoffman's proposed experiment: according to Mach's conjecture, the plane of oscillation of a Foucault pendulum (for simplicity here at the North Pole) should be 'dragged' by the Earth's rotation, and consequently should not, relative to the distant stars, complete a rotation in 24 hours, lagging slightly behind.

proposed to use the Earth as the 'bucket', and a Foucault pendulum as the water (let's put it at one pole for simplicity (Fig. 7). If Newton is right, the rotation of the Earth should have no influence on the plane of oscillation of the pendulum, which should remain fixed with respect to absolute space. But if Mach is right, the large mass of the Earth should 'drag' the inertial frame of reference of the pendulum, making it rotate with it very slowly. One would then see that, relative to the stars, the pendulum would not complete a circle in 24 hours, but would take slightly longer.

This experiment, as it was conceived, had no hope of succeeding. But it was actually successfully performed in the early 2000's, in a slightly modified version. One just needs the longest Earth-bound pendula that humans ever built: artificial satellites. The Lageos and Gravity probe B satellites did the job, and they detected a rotation of their orbital plane due to Earth's 'frame-dragging' effect. After more than a century, Mach has been proved right and Newton's absolute space ruled out.

[12] One possibility was that the Earth's rotation speed was decreasing, givng rise to a spurious apparent acceleration of the Moon. However, it was also suggested that gravity could be absorbed by matter. Then during eclipses of the Moon the presence of the Earth between it and the Sun would reduce the gravitational force acting on the Moon and could explain the discrepancy. This possibility was only ruled out definitively by observations in 1939, which showed that the planets also exhibited the same anomalous acceleration as the Moon. Meanwhile, the creation of General Relativity in 1915 by Einstein had explained the longstanding anomalous advance of Mercury's perihelion as due to failure of Newton's gravitational law.

[13] In fact, Hofmann's proposal had been anticipated in 1896 by the brothers Benedict and Immanuel Friedlaender (see [27]), one of whom actually did experiments with flywheels to text Mach's idea. I discuss Hoffmann's proposal because, more realistically, it uses rotation of the Earth.

3.4 Poincaré's principle

The rather vague idea that Mach sketched in his *Mechanics* was not sufficiently precise to make an actual principle out of it. It has nonetheless been called 'Mach's principle' (Einstein coined the expression). Einstein, despite being a strong advocate of Mach's principle for years, never found a satisfactory formulation of it, and towards the end of his life even disowned it, claiming that it had been made obsolete by the advent of field theory.[14]

In fact, a precise formulation of Mach's principle had been there in front of Einstein's eyes all the time, but nobody recognized it for what it was until Barbour and Bertotti did in 1982 [29]. This formulation is due to Henri Poincaré, in his *Science and Hypothesis* (1902). The delay in the identification of this important contribution is due to the fact that Poincaré himself never thought of it as a precise formulation of Mach's principle. In fact, although Poicaré can hardly have been unfamiliar with Mach's work, he did not cite it in *Science and Hypothesis*.

What Poincaré did ask was this: "What precise defect, if any, arises in Newton's mechanics from his use of absolute space?". The answer he gave can be understood in the light of our discussion of Tait's note: from *observable* initial configurations and their *first derivatives* alone one cannot predict the future evolution of the system.

The cause of this is, according to Poincaré, angular momentum. There is no way one can deduce the *total angular momentum of the system one is considering* from the observable initial data r_{ab} and their first derivatives alone. This can be achieved by looking at the second derivatives, as was demonstrated by Lagrange in 1772 [30] for the 3-body problem, but this remedy is unnatural, especially for the N-body problem when N is large: one needs only 3 out of the $3N - 6$ second derivatives.

Poincaré found this situation, in his words, "repugnant", but had to accept the observed presence of a total angular momentum of the Solar System, and renounce to further his critique. Interestingly, it didn't occur to Poincaré that the Solar System isn't the whole Universe, it is instead a rather small part of it as was already obvious in 1902.

Barbour and Bertotti therefore proposed what they called *Poincaré's principle*: The law of the Universe as a whole should be such that for it specification of initial inter-particle separations r_{ab} and their rates of change should determine the evolution uniquely. There is a natural generalization of this law to dynamical geometry.

[14] In his Autobiographical Notes [28], p.27, Einstein declares: "Mach conjectures that in a truly rational theory inertia would have to depend upon the interaction of the masses, precisely as was true for Newton's other forces, a conception that for a long time I considered in principle the correct one. It presupposes implicitly, however, that the basic theory should be of the general type of Newton's mechanics: masses and their interaction as the original concepts. Such an attempt at a resolution does not fit into a consistent field theory, as will be immediately recognized."

Further reading: Mach's "The Science of Mechanics" [24], Barbour's essay "The nature of time" [25], Poincaré's "Science and Hypothesis" [31]. The great conference proceedings "Mach's Principle: From Newton's bucket to quantum gravity" [27].

Part II

Relational Particle Dynamics

4 Barbour and Bertotti's discovery of best matching

As I said, Julian Barbour and Bruno Bertotti in 1982 [29] recognized that Poincaré had effectively given a mathematically precise formulation of Mach's principle and dubbed it the *Poincaré Principle* (Barbour now calls it the Mach–Poincaré Principle:)

> *Physical (or relational) initial configurations and their first derivatives alone should determine uniquely the future evolution of the system.*

In the paper [29] Barbour and Bertotti implemented this principle through what they called the intrinsic derivative and Barbour now calls *best matching*, which allows one to establish a notion of equilocality – to say when two points are at the same position at different instants of time when only relational data are available.

4.1 Best matching: intuitive approach

The basic idea is this: say that you're an astronomer who is given two pictures of three stars, taken some days apart (assume, for simplicity, that the stars are fixed on a plane orthogonal to the line of sight; one could ascertain that, for example, by measuring their redshifts). You're not given any information regarding the orientation of the camera at the time the two pictures were taken. The task is to find an *intrinsic* measure of change between the two pictures which does not depend on the change in orientation of the camera.

One is obviously only given the relative separations between the stars $r_{12}(t)$, $r_{23}(t)$, $r_{31}(t)$ at the two instants $t = t_i, t_f$. The task is to identify a Cartesian representation of the three particle positions $\mathbf{r}_a(t) = (x_a, y_a, z_a)(t) \in \mathbb{R}^3$, $a = 1, 2, 3$, at $t = t_i, t_f$ such that

$$\|\mathbf{r}_a(t_i) - \mathbf{r}_b(t_i)\| = r_{ab}(t_i)\,, \quad \|\mathbf{r}_a(t_f) - \mathbf{r}_b(t_f)\| = r_{ab}(t_f)\,, \quad \forall\, a, b = 1, 2, 3, \tag{2}$$

where $\|\mathbf{r}_a\| = \|(x_a, y_a, z_a)\| = |x_a|^2 + |y_a|^2 + |z_a|^2$.

Now make a tentative choice of Cartesian representation, $\mathbf{r}_a(t_i) = r^i_a$ and $\mathbf{r}_a(t_f) = r^f_a$. Notice that the Cartesian representation of the configuration at

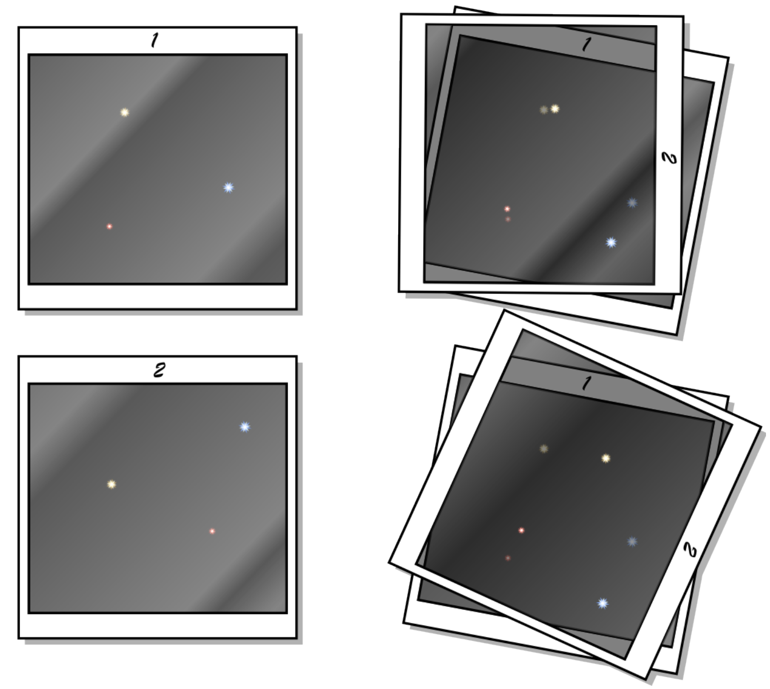

Figure 8: Different notions of equilocality: the two pictures on the left represent the same triple star system at different times. The relative positions of the stars have changed, and the astronomers are presented with the challenge of superposing picture 2 on picture 1 without having any reference background. On the right two such choices, corresponding to different definitions of equilocality, are depicted.

any instant consists of three arbitrary vectors of $\mathbb{R}^3$, which can be repackaged into a single vector of $\mathbb{R}^9$, because of course $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^9$. So let's interpret the configurations at the initial and final instants as two vectors of $\mathbb{R}^9$:

$$\begin{aligned} q^i &= \oplus^3_{a=1}\, \mathbf{r}^i_a = (\mathbf{r}^i_1, \mathbf{r}^i_2, \mathbf{r}^i_3) = (x^i_1, y^i_1, z^i_1, x^i_2, y^i_2, z^i_2, x^i_3, y^i_3, z^i_3)\,, \\ q^f &= \oplus^3_{a=1}\, \mathbf{r}^i_a = (\mathbf{r}^f_1, \mathbf{r}^f_2, \mathbf{r}^f_3) = (x^f_1, y^f_1, z^f_1, x^f_2, y^f_2, z^f_2, x^f_3, y^f_3, z^f_3)\,. \end{aligned} \tag{3}$$

We need a criterion to judge the 'distance' between q^i and q^f. There is a natural notion of distance on $\mathbb{R}^9$ given by the Euclidean distance $\mathrm{d} : \mathbb{R}^9 \times \mathbb{R}^9 \to \mathbb{R}$ which is just the square root of the sum of the square of the difference between each component of the two $\mathbb{R}^9$-vectors:

$$\mathrm{d}\left(q^i q^f\right) = \left[\sum_{\alpha=1}^{9}(q^i_\alpha - q^f_\alpha)^2\right]^{\frac{1}{2}} = \left(\sum_{a=1}^{3}\|\mathbf{r}^i_a - \mathbf{r}^f_a\|^2\right)^{\frac{1}{2}} . \tag{4}$$

This expression depends both on the intrinsic change of configuration between the two triangles and on the relative placement of picture 2 wrt picture 1. We can remove the latter dependence by trying all possible placements and finding the one that minimizes (4). In other words we have to find $q^{\text{BM}} = \oplus_{a=1}^3 \mathbf{r}^{\text{BM}}_a$ such that

$$\mathrm{d}\left(q^i, q^{\text{BM}}\right) = \inf_{q'} \mathrm{d}\left(q^i, q'\right) , \quad \|\mathbf{r}^{\text{BM}}_a - \mathbf{r}^{\text{BM}}_b\| = r_{ab}(t_f) , \tag{5}$$

(I chose to move the second picture, but obviously I could equivalently have moved the first one). In other words, we minimize with respect to transformations of the second triangle that keep its observable properties (the three $r_{ab}(t_f)$) unchanged. Those are the Euclidean transformations $\mathsf{Eucl}(3) = ISO(3) = SO(3) \ltimes \mathbb{R}^3$, which act on a single particle coordinate-vector as

$$\mathbf{r}_a \to \Omega\, \mathbf{r}_a + \boldsymbol{\theta} , \tag{6}$$

if $q = \mathbf{r}_1 \oplus \mathbf{r}_2 \oplus \mathbf{r}_3$, a Euclidean transformation will act identically on all three particles:

$$q \to T[q] = \oplus_{a=1}^3 \left(\Omega\, \mathbf{r}_a + \boldsymbol{\theta}\right) \tag{7}$$

where $\boldsymbol{\theta} \in \mathbb{R}^3$ and $\Omega \in SO(3)$. We can introduce the auxiliary variables Ω and $\boldsymbol{\theta}$ in order to directly perform the constrained variation to find q^{BM} defined in (5):

$$\mathrm{d}_{\text{BM}}\left(q^i, q^f\right) = \inf_T \mathrm{d}(q^i, T[q^f]) = \inf_{\Omega,\boldsymbol{\theta}}\left(\sum_{a=1}^{3}\|\mathbf{r}^i_a - \Omega\, \mathbf{r}^f_a - \boldsymbol{\theta}\|^2\right)^{\frac{1}{2}} . \tag{8}$$

The result is a notion of distance, called *best-matched distance*, that depends only on relational data. The quantity (8) is in fact invariant under Euclidean transformations of either of the two vectors q^i and q^f:

$$\begin{aligned}
\mathrm{d}_{\text{BM}}\left(T'[q^i]\, , q^f\right) &= \inf_{\Omega\, ,\boldsymbol{\theta}}\left(\sum_{a=1}^{3}\|\Omega'\, \mathbf{r}^i_a + \boldsymbol{\theta}' - \Omega\, \mathbf{r}^f_a - \boldsymbol{\theta}\|^2\right)^{\frac{1}{2}} \\
&= \inf_{\Omega\, ,\boldsymbol{\theta}}\left(\sum_{a=1}^{3}\|\mathbf{r}^i_a - (\Omega')^{-1}\Omega\, \mathbf{r}^f_a - (\Omega')^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}')\|^2\right)^{\frac{1}{2}} \\
&= \inf_{\Omega''\, ,\boldsymbol{\theta}''}\left(\sum_{a=1}^{3}\|\mathbf{r}^i_a - \Omega''\, \mathbf{r}^f_a - \boldsymbol{\theta}''\|^2\right)^{\frac{1}{2}} = \mathrm{d}_{\text{BM}}(q^i\, , q^f) ,
\end{aligned} \tag{9}$$

and similarly $\mathrm{d}_{\text{BM}}\left(q^i\, , T'[q^f]\right) = \mathrm{d}_{\text{BM}}\left(q^i\, , q^f\right)$.

For the simple problem of three stars the best-matching condition can be solved explicitly as a variational problem. Start with the translations, and consider a variation $\boldsymbol{\theta} \to \boldsymbol{\theta} + \delta\boldsymbol{\theta}$ that leaves the squared distance $\mathrm{d}^2\left(q^i\, , T[q^f]\right)$ stationary (there's no need to vary the square-root, as the square root is a monotonic function)

$$\frac{\delta\, \mathrm{d}^2\left(q^i\, , T[q^f]\right)}{\delta\boldsymbol{\theta}} = 2\sum_{a=1}^{3}(\mathbf{r}^i_a - \Omega\, \mathbf{r}^f_a) - 6\,\boldsymbol{\theta} = 0 , \tag{10}$$

which gives us the best-matching condition

$$\boldsymbol{\theta}^{\text{BM}} = \frac{1}{3}\sum_{a=1}^{3}(\mathbf{r}^i_a - \Omega\, \mathbf{r}^f_a) , \tag{11}$$

this condition establishes that to best-match wrt translations we just have to subtract the barycentric coordinates from both $\mathbf{r}^i_a$ and $\mathbf{r}^f_a$. In other words, we have to make the barycenters of the two triangles coincide:

$$\mathrm{d}_{\text{BM}}\left(q^i\, , q^f\right) = \inf_{\Omega}\left(\sum_{a=1}^{3}\|\Delta\mathbf{r}^i_a - \Omega\, \Delta\mathbf{r}^f_a\|^2\right)^{\frac{1}{2}} , \tag{12}$$

where $\Delta\mathbf{r}_a = \mathbf{r}_a - \frac{1}{3}\sum_a \mathbf{r}_a$ are the barycentric coordinates.

Now we need to vary with respect to rotations. Taking $\Omega \to \Omega + \delta\Omega$ (in the sense of varying independently all the 9 components of Ω) is wrong, because we have to make sure the variation keeps the matrix an element of $SO(3)$. Imposition of this condition,

$$(\Omega + \delta\Omega)(\Omega + \delta\Omega)^{\text{T}} = \mathrm{I} + \Omega\, \delta\Omega^{\text{T}} + \delta\Omega\, \Omega^{\text{T}} = \mathrm{I} , \tag{13}$$

translates into the requirement of antisymmetry of the matrix $\delta\Omega\, \Omega^{\text{T}}$:

$$\delta\Omega\, \Omega^{\text{T}} = -\left(\delta\Omega\, \Omega^{\text{T}}\right)^{\text{T}} . \tag{14}$$

In 3D any antisymmetric matrix can be written in terms of the 'vector-product' operator with a vector $\delta\boldsymbol{\omega}$

$$\delta\Omega\, \Omega^{\text{T}} = \delta\boldsymbol{\omega}\times , \quad \Rightarrow \quad \delta\Omega = \delta\boldsymbol{\omega} \times \Omega , \tag{15}$$

where $\boldsymbol{v}\times$ is to be understood as the matrix of components $\epsilon_{ijk}v_k$. So our variation takes the form $\Omega \to (\mathrm{I} + \delta\boldsymbol{\omega}\times)\,\Omega$, where $\delta\boldsymbol{\omega}$ is an infinitesimal vector that is parallel to the rotation axis. Imposing stationarity of (12) wrt $\delta\boldsymbol{\omega}$ variations, we get

$$\sum_{a=1}^{3} \|\Delta \mathbf{r}^i_a - (\mathrm{I} + \delta\boldsymbol{\omega}\times)\Omega\, \Delta \mathbf{r}^f_a\|^2 = \sum_{a=1}^{3} \|\Delta \mathbf{r}^i_a - \Omega\, \Delta \mathbf{r}^f_a\|^2$$
$$+2\,\delta\boldsymbol{\omega} \cdot \sum_{a=1}^{3} (\Omega\, \Delta \mathbf{r}^f_a) \times \left(\Delta \mathbf{r}^i_a - \Omega\, \Delta \mathbf{r}^f_a\right) = 2\,\delta\boldsymbol{\omega} \cdot \sum_{a=1}^{3} (\Omega\, \Delta \mathbf{r}^f_a) \times \Delta \mathbf{r}^i_a = 0\,. \tag{16}$$

The above equation would be hard to solve were it not for a simplification: three particles always lie on a plane. Therefore the infimum of $\mathrm{d}\left(q^i\,,\, T[q^f]\right)$ will necessarily be found among those choices of Ω that make the two triangles coplanar. This is easily understood, because all the coplanar positionings of the two triangles are always local minima of $\mathrm{d}\left(q^i\,,\, T[q^f]\right)$ under variations $\delta\boldsymbol{\omega}$ that break the coplanarity. Say that we choose the axes in such a way that $\Delta \mathbf{r}^i_a$ lies on the x, y plane and start with an Ω that keeps $\Omega\, \Delta \mathbf{r}^f_a$ on the same plane; then the quantity (16) is obviously parallel to the z plane. Therefore the variations of $\mathrm{d}\left(q^i\,,\, T[q^f]\right)$ in the x and y directions vanish.

So let's assume that $\Delta \mathbf{r}^i_a = (\Delta x^i_a, \Delta y^i_a, 0)$ and $\Delta \mathbf{r}^f_a = (\Delta x^f_a, \Delta y^f_a, 0)$, and take for Ω a rotation in the x, y plane. Eq. (16) becomes

$$\sum_{a=1}^{3} \left[\Delta x^i_a \left(\cos\phi\, \Delta y^f_a + \sin\phi\, \Delta x^f_a\right) - \Delta y^i_a \left(\cos\phi\, \Delta x^f_a - \sin\phi\, \Delta y^f_a\right)\right] = 0\,, \tag{17}$$

which simplifies to

$$\sum_{a=1}^{3} \|\Delta \boldsymbol{r}^i_a \times \Delta \boldsymbol{r}^f_a\| \cos\phi + \sum_{a=1}^{3} \Delta \boldsymbol{r}^i_a \cdot \Delta \boldsymbol{r}^f_a\, \sin\phi = 0\,, \tag{18}$$

which is easily solved

$$\phi_{\text{sol}} = \arctan\left(\frac{\sum_{a=1}^{3} \|\Delta \boldsymbol{r}^f_a \times \Delta \boldsymbol{r}^i_a\|}{\sum_{b=1}^{3} \Delta \boldsymbol{r}^i_b \cdot \Delta \boldsymbol{r}^f_b}\right)\,. \tag{19}$$

The expression above transforms in a simple way under separate rotations of $\Delta \boldsymbol{r}^f_a$ and $\Delta \boldsymbol{r}^i_a$: one can verify that (left as an exercise) under a rotation $\Omega(\alpha)$ in the $x - y$ plane:

$$\begin{aligned} \Delta \boldsymbol{r}^i_a &\to \Omega(\alpha) \Delta \boldsymbol{r}^i_a\,, \qquad & \phi_{\text{sol}} &\to \phi_{\text{sol}} - \alpha\,, \\ \Delta \boldsymbol{r}^f_a &\to \Omega(\alpha) \Delta \boldsymbol{r}_a\,, \qquad & \phi_{\text{sol}} &\to \phi_{\text{sol}} + \alpha\,, \end{aligned} \tag{20}$$

which implies that the best-matched distance

$$\mathrm{d}_{\text{BM}}\left(q^i\,,\, q^f\right) = \left(\sum_{a=1}^{3} \|\Delta \mathbf{r}^i_a - \Omega(\phi_{\text{sol}})\, \Delta \mathbf{r}^f_a\|^2\right)^{\frac{1}{2}}\,, \tag{21}$$

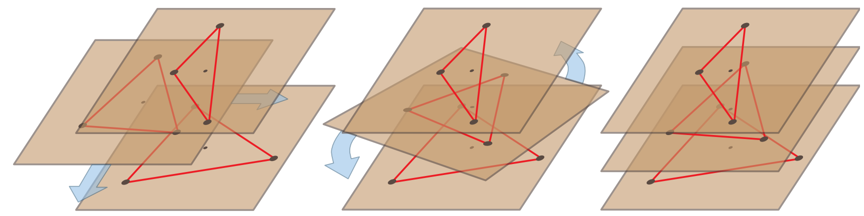

Figure 9: Horizontal stacking: in the picture a *stacking* of three-body configurations is represented. The arbitrarily chosen stacking on the left is best matched (blue arrows) by translations so as to bring the barycenters to coincidence, after which rotational best matching eliminates residual arbitrary relative rotation.

is invariant under separate rotations and translations of q^i or q^f.

We have obtained an expression that allows us to measure the amount of *intrinsic* change between the two configurations – change that is not due to an overall translation or rotation. This is the essence of best matching. And it is deeply connected to the theory of connections on principal fibre bundles: it defines a *horizontal derivative.*

Now, the distance d_{BM} we found is not very physical. First of all, it doesn't take into account the masses of the particles. If we take two configurations and move around just one particle, the d_{BM} between the two configurations changes independently of the mass of the particle even if we move just an atom while the other particles have stellar masses. This can be easily corrected by weighting the original Euclidean distances with the masses of the particles:

$$\mathrm{d}\left(q^i, q^f\right) = \left(\sum_{a=1}^{3} m_a \|\mathbf{r}^i_a - \mathbf{r}^f_a\|^2\right)^{\frac{1}{2}}\,. \tag{22}$$

Moreover, to introduce forces, as we shall see, we would like to weight different relative configurations differently, but without giving different weights to configurations that are related by a global translation or rotation. We can do that by multiplying by a rotation- and translation-invariant function:

$$\mathrm{d}\left(q^i, q^f\right) = \left(U(r_{bc}) \sum_{a=1}^{3} m_a \|\mathbf{r}^i_a - \mathbf{r}^f_a\|^2\right)^{\frac{1}{2}}\,. \tag{23}$$

The idea is now to perform this best-matching procedure for a complete history. Take a series of snapshots of the kind depicted in Fig. 8, and represent them in an arbitrary way in Euclidean space, $q^k = \oplus_{a=1}^{3} \mathbf{r}^k_a$, $k = 1, 2, \ldots$. Then minimize $\mathrm{d}\left(T_k[q^k], T_{k+1}[q^{k+1}]\right)$ wrt Ω_k and $\boldsymbol{\theta}_k$ for all k. What we get is a *horizontal stacking* (by the blue arrows in Fig. 9) of all the configurations. In other words, a notion of *equilocality*: the dots in the successive snapshots of the third part of Fig. 9 can all be said 'to be in the same place'. This notion of position is not a pre-existing absolute structure but emerges out of the best matching.

Continuum limit

Now consider the continuum limit, in which we parametrize the snapshots with a continuous parameter s, with each successive snapshot separated by an infinitesimal $\mathrm{d}s$:

$$v^k = v(s)\,, \qquad v^{k+1} = v(s+\mathrm{d}s)\,, \qquad v = \mathbf{r}_a\,,\ \Omega\,,\ \boldsymbol{\theta}\,, \tag{24}$$

Then the squared distance between two successive configurations is infinitesimal,

$$\begin{aligned} &\mathrm{d}^2\left(T_k[q^k], T_{k+1}[q^{k+1}]\right) = \\ &U(r_{bc}) \sum_{a=1}^{3} m_a \|\Omega(s+\mathrm{d}s)\,\mathbf{r}_a(s+\mathrm{d}s) + \boldsymbol{\theta}(s+\mathrm{d}s) - \Omega(s)\,\mathbf{r}_a(s) - \boldsymbol{\theta}(s)\|^2 \\ &\sim U(r_{bc}) \sum_{a=1}^{3} m_a \|\Omega\,\mathrm{d}\mathbf{r}_a + \mathrm{d}\Omega\,\mathbf{r}_a + \mathrm{d}\boldsymbol{\theta}\|^2\,, \qquad \mathrm{d}v = \frac{\partial v}{\partial s}\,\mathrm{d}s\,. \end{aligned} \tag{25}$$

For $\mathrm{d}\Omega$ too, we can repeat the argument for obtaining a variation of an $SO(3)$ matrix that remains within the group, and we get

$$U(r_{bc}) \sum_{a=1}^{3} m_a \|\Omega\,\mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega} \times \Omega\,\mathbf{r}_a + \mathrm{d}\boldsymbol{\theta}\|^2\,. \tag{26}$$

We can get rid of the dependence on Ω by exploiting the invariance of the scalar product under rotations:

$$\|\Omega\,\mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega} \times \Omega\,\mathbf{r}_a + \mathrm{d}\boldsymbol{\theta}\|^2 = \|\,\mathrm{d}\mathbf{r}_a + \Omega^{-1}\mathrm{d}\boldsymbol{\omega} \times \Omega\,\mathbf{r}_a + \Omega^{-1}\mathrm{d}\boldsymbol{\theta}\|^2 \tag{27}$$

Here, $\Omega^{-1}\mathrm{d}\boldsymbol{\omega} \times \Omega$ and $\Omega^{-1}\mathrm{d}\boldsymbol{\theta}$ are just the adjoint action of $SO(3)$ on $ISO(3)$. Minimizing wrt $\mathrm{d}\boldsymbol{\omega}$ and $\mathrm{d}\boldsymbol{\theta}$ makes this action irrelevant:

$$\mathrm{d}_{\text{BM}}\mathcal{L}^2 = \inf_{\mathrm{d}\boldsymbol{\omega},\mathrm{d}\boldsymbol{\theta}} U(r_{bc}) \sum_{a=1}^{3} m_a \|\mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega} \times \mathbf{r}_a + \mathrm{d}\boldsymbol{\theta}\|^2\,. \tag{28}$$

The last expression can be integrated over a parametrized path $\mathbf{r}_a(s)$,

$$\int \mathrm{d}_{\text{BM}}\mathcal{L} = \inf_{\mathrm{d}\boldsymbol{\omega},\mathrm{d}\boldsymbol{\theta}} \int \mathrm{d}s \left(U(r_{bc}) \sum_{a=1}^{3} m_a \left\| \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s} + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}s} \times \mathbf{r}_a + \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}s} \right\|^2 \right)^{\frac{1}{2}}\,. \tag{29}$$

We obtain a notion of length of the path that measures only the intrinsic, physical change that occurs along the path. The above expression is in fact invariant under *local* (s-dependent) $ISO(3)$ transformations

$$\mathbf{r}_a(s) \to \Omega(s)\,\mathbf{r}_a(s) + \boldsymbol{\theta}(s)\,. \tag{30}$$

This kind of object is what is needed to define a variational principle for relational physics. It realizes the kind of foundations for dynamics implied by Leibniz's criticism of Newton's concepts of absolute space and time. No wonder that, at the time, he was unable to make this precise.

Generating dynamics with best-matching

One can use this measure of intrinsic change to assign a numerical value (weight) to any curve in the 'extended configuration space' (the space of Cartesian representations of N particles), and thereby define an action to be minimized by the dynamical solutions. However, such an action principle is only capable of generating solutions with zero total momentum and angular momentum, as will be shown below. Therefore such a law cannot be used to describe a general N-particle system (like billiard balls on a table, or a gas in a box): it should rather be used to describe a complete universe.

It is in this sense that the novel foundation of dynamics I am describing satisfies Mach's principle and solves the puzzle of Newton's bucket (in the restricted case of a universe composed of point particles interacting with instantaneous potentials): the dynamical law is such that it can only contemplate a universe with zero total angular momentum. In Newtonian dynamics, on the contrary, angular momentum is a constant of motion that is freely specifiable through the initial conditions of our solution. The law that the total angular momentum of the universe must be zero solves the problem of Newton's bucket in the following way: one makes a (small) error in assuming that the reference frame defined by the fixed stars is inertial. If the total angular momentum is zero, and the Earth is rotating wrt the fixed stars, that reference frame cannot be inertial. This is due to the fact that the total angular momentum of the rest of the universe must be equal and opposite to that of the Earth. The reference frame of the fixed stars must be rotating very slowly (because the stars greatly overweigh the Earth), and a truly inertial frame would be rotating both wrt Earth and wrt the fixed stars. Then one would, in principle, see exactly what Wenzel Hoffman proposed: the plane of a pendulum at the north pole would rotate around the axis of the Earth at a slightly slower speed than the Earth itself. This effect is, of course, impossible to detect as it is too small. But this illustrates how a relational dynamics dissolves the puzzle of Newton's bucket.

Further reading: Barbour and Bertotti's 1982 seminal paper [29].

5 Best matching: technical details

I'll now make the ideas introduced above slightly more precise. First of all, we need to specify the various *configuration spaces* we're dealing with. The largest – and simplest – of them all is the *extended configuration space*, or *Cartesian space* $\mathsf{Q}^N = \mathbb{R}^{3N}$.[15] It is the space of Cartesian representations of N bodies, and the mathematical embodiment of Newton's absolute space. Q^N has a Euclidean metric on it, called the *kinetic metric*:

$$\mathrm{d}\mathcal{L}^2_{\text{kin}} = \sum_{a=1}^{N} m_a \, \mathrm{d}\mathbf{r}_a \cdot \mathrm{d}\mathbf{r}_a \,. \tag{31}$$

Then there is the *relative configuration space* $\mathsf{Q}^N_R = \mathsf{Q}^N/\mathsf{Eucl}(3)$, which is just the quotient of Q^N by the Euclidean group of rigid translations and rotations. Finally, if we insist that only ratios and angles have objective reality, we must further quotient by scale transformations:

$$\mathbf{r}_a \to \phi \, \mathbf{r}_a \quad \phi > 0 \,, \tag{32}$$

which, together with the Euclidean group, make the *similarity group* $\mathsf{Sim}(3) = \mathbb{R}^+ \ltimes \mathsf{Eucl}(3)$. We will call this last quotient the *shape space* of N particles, $\mathsf{S}^N = \mathsf{Q}^N/\mathsf{Sim}(3)$. However, since gauge-fixing or reducing wrt rotations is hard except in some simple cases, we also consider the configuration space obtained by quotienting just wrt translations and dilatations, $\mathsf{PS}^N = \mathsf{Q}^N/\mathbb{R}^+ \ltimes \mathbb{R}^3$. We call this *pre-shape space*.

Reduced conf. space	G	$B = \mathsf{Q}^N/G$
Relative conf. space	$\mathsf{Eucl}(3) = ISO(3)$	Q^N_R
Pre-shape space	$\mathbb{R}^+ \ltimes \mathbb{R}^3$	PS^N
Shape space	$\mathsf{Sim}(3) = \mathbb{R}^+ \ltimes ISO(3)$	S^N

Table 1: Different kinds of configuration space reductions of Q^N.

Both mathematically and conceptually, it is important that the various reduced (quotient) spaces are not subspaces of the space from which they are obtained by reduction but distinct spaces.

[15]For this and the other configuration spaces considered below, there is in fact a distinct space for each set of masses (or, better, mass ratios) of the particles.

Principal fibre bundles

The groups acting on Q^N endow it with the structure of a *principal G-bundle* [32] (this actually holds only for the regular configurations in Q^N, see below). Let's call the principal G-bundle P and the group G. The reduced configuration space plays the role of the *base space*, which in a principal bundle is the quotient space $B = P/G$. The fibres are homeomorphic to the group G.[16] As relationists, our prime interest is in the base space B, which we regard as the space of physically distinct configurations, but it is only defined through the quotienting process, and this poses problems: for example, B inherits P's structure of a smooth manifold only if the group G acts *freely* on P, which means that there are no points in P that are left invariant by any other transformation than the identity. But we know that this isn't the case in Q^N: there are symmetric configurations (e.g., collinear states or total collisions) for which the action is not free. These regions represent special parts of B, akin to corners or edges, where smoothness fails.[17] Continuing a dynamical orbit after it crosses one of those points poses a challenge, but this is a technical, rather than conceptual, issue that I won't go into here.

Intuitively, a fibre bundle is a sort of generalization of a Cartesian product $B \times G$ where one erects a tower of different representations in Q^N above every single relative configuration or shape (see Fig. 10). The difference with a true Cartesian product is the lack of an origin. Obviously a point in, say, Q^3 cannot be uniquely represented as a point in S^3 (a triangle), together with a translation vector $\boldsymbol{\theta}$, an $SO(3)$ matrix Ω and a dilatation constant ϕ. This doesn't make sense because $\boldsymbol{\theta}$, Ω and ϕ are *transformations* that connect different Cartesian representations of the same triangle. This is where local sections (and 'trivializations') of the bundle enter: they provide a 'conventional' choice of origin for each triangle. This means that a section associates with each triangle in a neighbourhood of S^3 an oriented triangle with a definite size and position in $\mathbb{R}^3$. This is purely conventional: for example I could decide that all triangles go onto the x, y plane, with their barycenter at the origin, and the most acute of their three vertices goes on the x-axis at unit distance from the origin. Then I can represent any other element of Q^3 through the transformation that is needed to bring the 'reference' triangle to congruence with the desired one.

One is forced to define those sections/trivializations *locally*, that is only on a set of open neighbourhoods, because the sections have to be smooth (continuous and infinitely differentiable), and unless the bundle is 'trivial' no section can be

[16]The fibres and the group are just *homeomorphic* (meaning equivalent as topological spaces), not *isomorphic*, because the fibres lack an identity element, which is an essential part of the structure of a group. D. Wise alerted me that such a "group that has forgotten its identity" is called a *torsor*. See J. Baez' description of torsors [33].

[17]B becomes a *stratified manifold* [34].

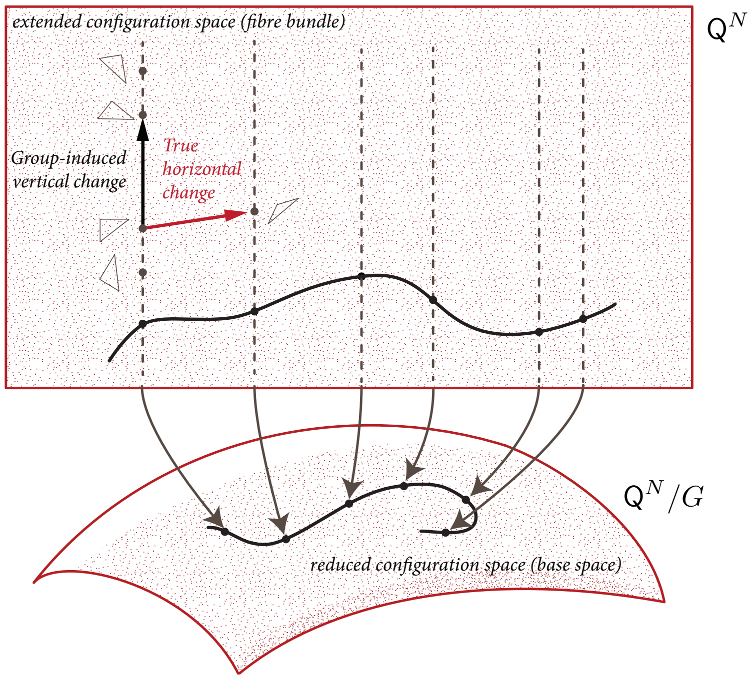

Figure 10: The fibre bundle structure of the three-body configuration space Q^3, with structure group $\mathsf{Sim}(3)$ and base manifold S^3 (the space of triangles). Vertical motion changes the *representation* of the triangle in Cartesian space, while horizontal motion changes the shape of the triangle (*e.g.,* its internal angles).

smooth everywhere.[18]

Physicists have a name of their own for local sections: *gauges*.

Connections

A principal G-bundle comes equipped with a natural distinction between *vertical* and *horizontal* directions. The first are defined as the subspace $V_p \subset T_pP$ of tangent vectors to P that are parallel to the orbits of G, and are related

[18]For example, the section I defined in the above example fails to be continuous when the smallest-angle vertex of the triangle changes: at that point I have an abrupt rotation of the representative triangle in Q^3. This might seem a quirk of the particular section that I chose, but it is instead an obstruction of topological nature: the topology of the Cartesian product $\mathsf{S}^3 \times \mathsf{Sim}(3)$ is different from that of Q^3. This is easily seen: $\mathsf{Q}^3 \sim \mathbb{R}^9$ is simply connected while the rotation group $SO(3)$ is not, and therefore neither is $\mathsf{S}^3 \times \mathsf{Sim}(3)$.

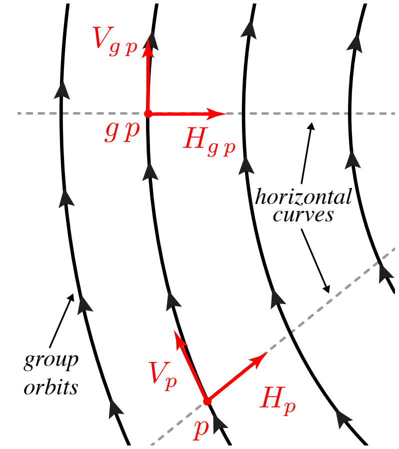

Figure 11: The G-invariance condition for a connection: if a horizontal curve through p is transformed by $g \in G$, then the tangent vector at $g\,p$ to the transformed curve must still be horizontal.

to unphysical, irrelevant or gauge motions. Horizontal directions represent physical, relevant, or true change (see Fig. 10), and they are just defined as the vector-space complement to V_p, that is $T_pP = V_p \oplus H_p$. It's clear that this definition is ambiguous: imagine that $T_pP = \mathbb{R}^3$, and V_p is one-dimensional. Then any plane that is not parallel to V_p is an equally good choice of H_p. If one has a metric on TP, then H_p can be defined as the (unique) *orthogonal* complement of V_p, but a principal bundle does not always come equipped with a metric on T_pP. This is what connections are introduced for. A connection on P is a smooth choice of horizontal subspaces in a neighbourhood of P.

A connection on P defines a (conventional) notion of horizontality, and consequently of *horizontal curves*: those curves whose tangent vectors are horizontal. This is the precise formalization of best-matched trajectories: they must be horizontal according to some connection on Q^N. Moreover a connection has to satisfy a compatibility condition with the action of the group G, which basically states that the G-action sends horizontal curves to horizontal curves (see Fig. 11).

As I said, if we have a metric on TP, then that defines a 'natural' notion of horizontality: $H_p = V_p^\perp$. In order for this to define a connection, the orthogonality condition defined by the metric has to be G-invariant,[19] so that a

[19]A G-invariant metric associates the same scalar product to the tangent vectors of two curves that intersect at a point and the corresponding tangent vectors to the curves transformed under G.

G-transformation sends curves that are orthogonal to V_p to curves orthogonal to $V_{g\,p}$. A particularly simple situation is that of a G-invariant metric: in that case the metric not only defines a connection, but all the horizontal curves are geodesics of that metric. In [29], Barbour and Bertotti exploited the following results to define a relational dynamical law, which I formulate in this way:

> ***Theorem:*** *If a metric on P is G-invariant, given a* sheet *in P lifted above a single curve in B, all the horizontal curves on that sheet minimize the free-end point length between the initial and final orbits. Moreover, all the horizontal curves on that sheet have the same length according to the G-invariant metric.*

The technical terms 'sheet' and 'free-end point' in this theorem will now be explained.

The two-stage variational procedure

Now I will describe the variational principle that realizes best-matching. The goal is to get an *action*, that is, a rule to associate a real number to each path on the base manifold P/G, *i.e.,* the reduced configuration space, be it Q^N_R or S^N. It is pretty clear, at this point, that only the simplest cases can be effectively worked out on P/G itself [35], since in general our only way to represent that space is redundantly through P. So what we aim for is an action principle on P which is G-invariant, so that it associates the same number to all paths that project to the same path in P/G and correspond therefore to the same physical solution.[20]

Stage I: Free-end-point variation

The following pictures illustrate the first stage of variation: our end-points are two points in P/G, where the physics resides. In P they map to two fibres, the two red lines in Fig. 12, which are two orbits of G.

1. First, take a trial curve in P (in black) between the two (red) fibres ending anywhere on them. It projects to a trial curve in B.

2. Then *lift* this curve in P to a sheet (in gray) in P using the group action. All the curves on this sheet correspond to the same *physical* curve in B.

[20] A note of warning is in order here: I am not trivially talking about an action that is invariant under a 'global' translation or rotation of the system. For that purpose the actions everybody is familiar with from basic physics courses are perfectly good. Here I'm talking about an action which is invariant under *time-dependent* transformations. In this sense the step from elementary action principles is perfectly analogous to that from a global to a local gauge symmetry in field theory. The only difference is that the 'locality' here is only in time, not in spacetime as in electromagnetism.

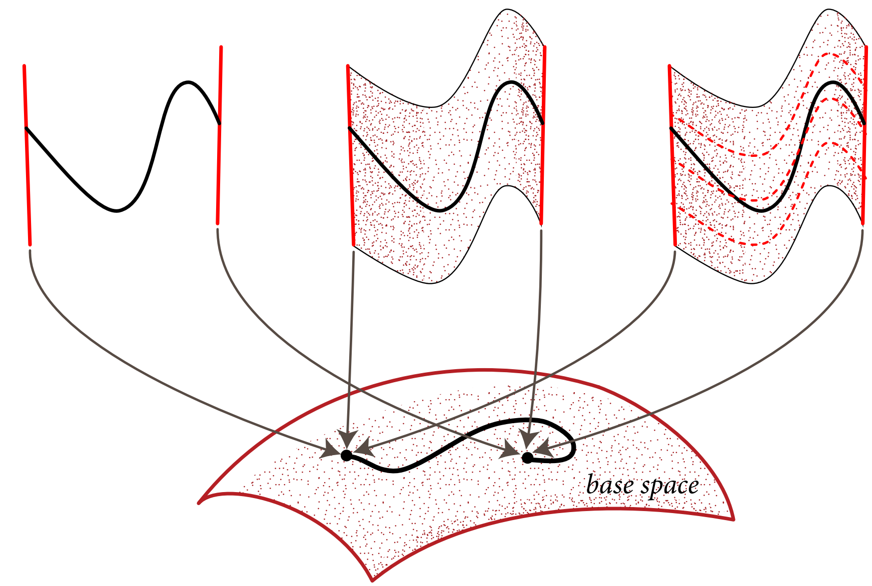

Figure 12: Stage I: Free-end-point variation on a sheet in P. Notice how this is all done in the bundle, P, and the base space B is abstractly defined as the collection of all gauge orbits (here represented as red lines).

3. Consider all the horizontal curves (red dashed) in the sheet with endpoints *anywhere* on the two red fibres. If the metric in P is G-invariant, these curves also minimize the arc-length from the first fibre to the second. This is obtained through a *free-end-point variation* (see below).

4. If the metric is G-invariant, all these horizontal curves will have the same length. We'll define the value of the action on the curve in B corresponding to the considered sheet as the length of the curve. In this way the action is G-invariant.

Stage II: Physical variation

Now we have an action associated with a *sheet* in P and consequently with a single path in P/G. We can now evaluate it on every possible path in P/G between the two endpoints, which means on every possible sheet in P between the two red fibres:

5 Consider all possible paths in P/G joining the two red fibres. Use the above rule to assign a value of the action to each of them. If two of them lie on the same sheet, they have the same action. The action is G-invariant.

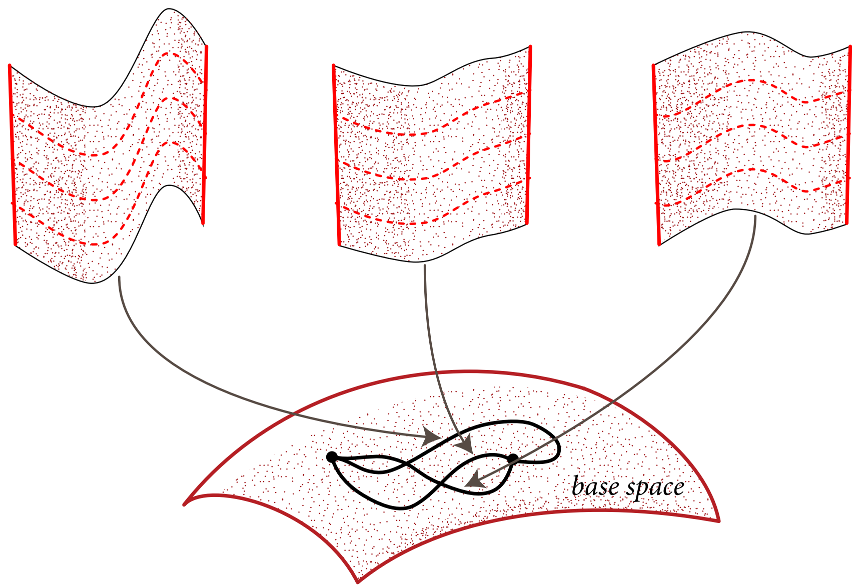

Figure 13: Stage II: Variation of the physical curve in P/G to find the one (on the right) that realizes the extremum between the fixed end points in P/G.

6 Minimizing with respect to this action will identify the physically realized sheet, and consequently a unique curve in P/G.

On this final sheet there is a $\dim(G)$-parameter family of horizontal curves. They are distinguished only by their initial positions on the fibre. These curves are not more physical than the others on the same sheet, but they realize a reference frame in which the equations of motion take the simplest form. In this sense they are special (Barbour and Bertotti introduced the notion of the *distinguished representation*). They give a preferred notion of equilocality (horizontal placement in Fig. 9).

Before showing this technique in action, let me comment on temporal relationalism.

5.1 Temporal relationalism: Jacobi's principle

So far, temporal relationalism found less space than spatial relationalism in this tutorial. It is time to introduce it. I will show now how to realize Mach's aphorism [24] "time is an abstraction at which we arrive through the changes of things" in a dynamical theory.

The necessary mathematics had actually been created by Jacobi in 1837, nearly 50 years before Mach wrote that sentence. Jacobi was not thinking about the abstraction of time from change – he seems to have been happy with Newton's concept of time. Rather, his aim was to give a mathematically correct formulation of Maupertuis's principle. Ever since its original statement, it had been assumed, above all by Euler and Lagrange, that all trial curves considered for comparison must correspond to the same total energy. The problem that Jacobi solved was the correct mathematical representation of such a condition. Euler and Lagrange had got the right answer using dubious mathematics.

Jacobi achieved his aim by reformulating Mapertuis's principle, for systems with a quadratic kinetic energy, in a 'timeless' form. His principle determines the 'true' (in the sense of physically realized) trajectories of a dynamical system *with one fixed value E of its total energy* as geodesics in Q^N, which are geometrical loci that do not depend on any particular parametrization.[21]

Jacobi's action is (integrating on a finite interval of the parameter $s \in [s_i, s_f]$)

$$S_{\rm J} = 2\int_{s_i}^{s_f} \mathrm{d}s\sqrt{(E-U)\,T_{\rm kin}}\,, \qquad T_{\rm kin} = \sum_{a=1}^{N} \frac{m_a}{2}\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s}\cdot\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s}\,, \tag{33}$$

where E is a constant (the total energy of the orbit we're interested in but here is to be regarded as a constant part of the potential, i.e., as part of the law that governs the system treated as an 'island universe'), $U = U(\mathbf{r}_a)$ is the potential energy and $T_{\rm kin}$ defined above is the kinetic energy. This expression is reparametrization-invariant

$$\mathrm{d}s \to \frac{\partial s}{\partial s'}\,\mathrm{d}s'\,, \qquad \sum_{a=1}^{N} m_a \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s}\cdot\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s} \to \left(\frac{\partial s'}{\partial s}\right)^2 \sum_{a=1}^{N} m_a \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s'}\cdot\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s'}\,, \tag{34}$$

(for reparametrizations s' that preserve the end-point value of the parameter, $s'(s_i) = s_i$ and $s'(s_f) = s_f$) thanks to the square-root form of the action. Jacobi's action is closely analogous to the expression for the arc-of length in a Riemannian manifold with metric $\mathrm{d}s^2 = g^{ij}\mathrm{d}x_i\,\mathrm{d}x_j$:

$$\int \mathrm{d}s\mathcal{L} = \int \mathrm{d}s\left(g^{ij}\frac{\mathrm{d}x_i}{\mathrm{d}s}\frac{\mathrm{d}x_j}{\mathrm{d}s}\right)^{\frac{1}{2}}\,, \tag{35}$$

and is precisely the same thing if our manifold is Q^N equipped with the metric

$$\mathrm{d}\mathcal{L}^2 = 4\,(E-U)\sum_{a=1}^{N}\frac{m_a}{2}\,\mathrm{d}\mathbf{r}_a\cdot\mathrm{d}\mathbf{r}_a\,, \tag{36}$$

[21] The same is true of a manifold, which doesn't depend on any particular coordinate system.

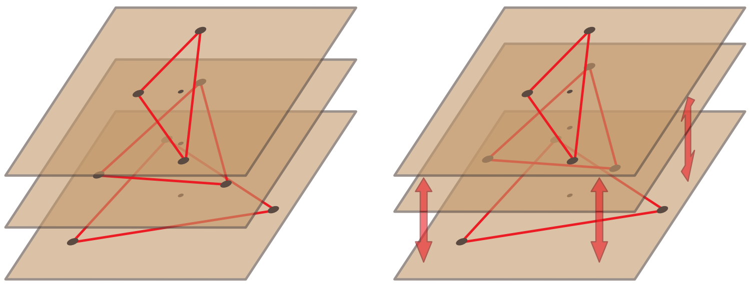

Figure 14: Vertical stacking: the same stacking of three-body configurations as in Fig. 9 can be moved *vertically* (red arrow) by a time reparametrization. While best matching fixes the horizontal stacking, ephemeris time fixes the vertical one.

which is conformally related to the kinetic metric by the factor $E - U$.

In the language of E. Anderson [35] we can rewrite the action in a manifestly *parametrization-irrelevant* form,

$$S_{\text{J}} = 2\int_{\mathbf{r}_a^i}^{\mathbf{r}_a^f} (E-U)^{\frac{1}{2}} \left(\sum \frac{m_a}{2}\, \mathrm{d}\mathbf{r}_a \cdot \mathrm{d}\mathbf{r}_a\right)^{\frac{1}{2}}, \tag{37}$$

where now $\mathbf{r}_a^i = \mathbf{r}_a(s_i)$ and $\mathbf{r}_a^f = \mathbf{r}_a(s_f)$ are the end-points. This is in fact the form in which Jacobi originally formulated his principle.

The ephemeris time

Jacobi's action gives rise to the following Euler-Lagrange equations:

$$m_a \frac{\mathrm{d}}{\mathrm{d}s}\left(\left(\frac{E-U}{T_{\text{kin}}}\right)^{\frac{1}{2}} \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}s}\right) = -\left(\frac{T_{\text{kin}}}{E-U}\right)^{\frac{1}{2}} \frac{\partial U}{\partial \mathbf{r}_a}, \tag{38}$$

which has a complicated form. But, as is always possible, now choose the parametrization in which

$$T_{\text{kin}} = E - U, \tag{39}$$

with the increment of the new parameter given by

$$\mathrm{d}t_{\text{eph}} = \mathrm{d}s\left(\frac{T_{\text{kin}}}{(E-U)}\right)^{\frac{1}{2}} = \left(\frac{\sum_{a=1}^N m_a \mathrm{d}\mathbf{r}_a \cdot \mathrm{d}\mathbf{r}_a}{2(E-U)}\right)^{\frac{1}{2}} = \frac{\mathrm{d}\mathcal{L}}{2(E-U)}. \tag{40}$$

Following the nomenclature of Edward Anderson, I call (40) *the differential of the instant*. It measures the accumulation of a distinguished 'duration' from one instant to another. The meaning of the subscript 'eph' will be explained shortly. In the parametrization (40), the equations of motion take the form

$$m_a \frac{\mathrm{d}^2\mathbf{r}_a}{\mathrm{d}t_{\text{eph}}^2} = -\frac{\partial U}{\partial \mathbf{r}_a}, \tag{41}$$

which is just Newton's equations for N particles interacting through the potential U. Note that Eq. (39), rewritten as $E = T_{\text{kin}} + V$, would normally be interpreted as the expression of energy conservation. But, in a Machian approach E for the Universe is to be interpreted, as I said, as a universal constant (like Einstein's cosmological constant), and then it, expressed explicitly through (40), becomes *the definition of time* or, better, *duration.*[22]

Indeed, Eq. (40) above is the closest thing to Mach's ideal we could imagine: an increment of time which is a sum of all the $\mathrm{d}\mathbf{r}_a$ in the particles' positions, weighted by (twice) the difference between total and potential energy. Out of a timeless theory like Jacobi's, a 'natural' parametrization emerges, which gives a notion of duration as a *distillation* of all the changes in the Universe [36]. In this time, the particles in the Universe, provided one identifies an inertial frame of reference (I'll show how in a moment), will move according to Newton's laws.

This is the place to explain the subscript 'eph' in $\mathrm{d}t_{\text{eph}}$ in (40). As I remarked above, in the 1890s astronomers found an apparent deviation from Newton's laws in an anomalous acceleration in the motion of the Moon [37]. One possible explanation was that this was nothing to do with a failure of Newton's laws but arose from the use of the Earth's rotation to measure time. This was only definitively confirmed in 1939, when it was shown that the planets exhibited the same anomaly. This confirmed that the rotating Earth doesn't tick a time in which Newton's laws are satisfied. The way out was to assume a more flexible notion of duration, in which the second is defined by an 'average' of the motions of the most prominent (and massive) objects in the solar system in such a way that Newton's laws are verified. In practice, to match the increasing accuracy of observations it is necessary to include more and more objects in this definition of time, which is called 'ephemeris time'.[23]

What is a clock?

In the physics community, there is a widespread misconception that to define clocks one needs to make reference to periodic phenomena, which provide a time standard because of their *isocronicity*. This tradition finds its origin in the imitation of Einstein [38], with his clocks made with mirrors and light rays.

[22] We ask "what's the time?", if we want to know which *instant* of time is it. But we also ask "How much time do we have?".

[23] From the Greek ἐφημερίς (ephēmeris) for "diary". Ephemerides are tables of the predicted positions of the celestial bodies over time.

But isocronicity of periodic phenomena per se is a circular argument (if you define the second as one period of a pendulum, then obviously the pendulum will always complete a period in exactly one second!). A more refined version of this idea is based on the hypothesis of the homogeneity (in time) of Nature: two phenomena that take place under identical conditions should take the same time. But also this is a fallacy: there are no two identical phenomena: "You can never step into the same river twice". Indeed, if two phenomena had exactly the same attributes, they would have to be identified (this Leibniz's principle of the *identity of the indiscernibles*).

The relationalist point of view is that the main defining property of good clocks (both natural and artificial) is that they *march in step*, and therefore they are useful for keeping appointments [25]. The fact that we (on Earth) can usefully make reference to an ever-flowing, ubiquitous notion of time is made possible by the way the objects in the world behave, with a lot of regularities. After Einstein and his relativity of simultaneity we cannot easily talk about a universal notion of a present instant,[24] but the huge degree of regularity that our Universe exhibits is by no means diminished: it just translates into the fact that we all can usefully make reference to a unique spacetime, and all local measurements of clocks (and distances, see below) are mutually consistent with a notion of spacetime. The simplest and prototypical example of a relationalist clock is Tait's *inertial clock*, which is the position of particle 2 in his solution to the Scholium problem (cfr. Sec. 3.1).

5.2 Best-matching 'in action'

Stage I: Free-end-point variation

Assume that we have a G-invariant metric $(\mathrm{d}\mathbf{r}_a, \mathrm{d}\mathbf{r}'_b)_{\mathrm{G}} = \sum_{a,b} \mathrm{M}^{ab}\, \mathrm{d}\mathbf{r}_a \cdot \mathrm{d}\mathbf{r}'_b$ on TP.[25] In accordance with Stage I of the best-matching procedure, we start with a generic path $\mathbf{r}_a(s) : [s_1, s_2] \to P$, then lift it *locally* along the fibres with the group action

$$\mathbf{r}_a(s) \to O(s)\,\mathbf{r}_a(s)\,, \tag{42}$$

and look for the horizontal paths. If the metric is G-invariant, those paths will minimize the length defined by the metric

$$\mathrm{d}\mathcal{L}_{\text{bare}} = \sqrt{(\mathrm{d}\mathbf{r}_a(s), \mathrm{d}\mathbf{r}_a(s))_{\mathrm{G}}} = \|\mathrm{d}\mathbf{r}_a(s)\|_{\mathrm{G}}\,, \tag{43}$$

so we *we only need to vary the group elements:*

$$S_{\text{BM}} = \inf_{O(s)} \int \mathrm{d}\mathcal{L} = \inf_{O(s)} \int \|\mathrm{d}\,(O(s)\,\mathbf{r}_a(s))\|_{\mathrm{G}}\,, \tag{44}$$

[24] Here 'easily' anticipates that there is a sense in which Shape Dynamics defines a universal present.

[25] Here I mean G-invariant in the sense described in footnote 19.

but keeping the end points free. This last requirement is of paramount importance. One normally takes the variation while keeping the end points fixed, which makes it possible to discard some boundary terms and obtain the Euler–Lagrange equations. Here variation of the end points along the fibres corresponds to unphysical motions, so nothing allows us to keep them fixed. The G-invariance of the metric makes it possible to rewrite the variational principle as

$$S_{\text{BM}} = \inf_{O(s)} \int \left\|\mathrm{d}\mathbf{r}_a + O^{-1}\mathrm{d}O\,\mathbf{r}_a\right\|_{\mathrm{G}}\,. \tag{45}$$

If O is a matrix representation of a Lie group, the expression $\mathrm{d}\epsilon = O^{-1}\,\mathrm{d}O$ is the differential of a representation of the corresponding Lie algebra $O(s) = \exp\,\epsilon(s)$. We can therefore replace $O(s)$ by an s-dependent Lie algebra element $\epsilon(s)$:

$$S_{\text{BM}} = \inf_{\epsilon(s)} \int \|\mathrm{d}\mathbf{r}_a + \mathrm{d}\epsilon\,\mathbf{r}_a\|_{\mathrm{G}}\,. \tag{46}$$

The last expression looks like a covariant differential, which we can call the *best-matching differential* [35]

$$\mathfrak{D}\mathbf{r}_a = \mathrm{d}\mathbf{r}_a + \mathrm{d}\epsilon\,\mathbf{r}_a\,. \tag{47}$$

The *free-end-point* Euler–Lagrange equations (see Appendix B.2 for their derivation) for (46) give

$$\mathrm{d}\left(\frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon}\right) = \frac{\delta\mathrm{d}\mathcal{L}}{\delta\epsilon}\,, \qquad \left.\frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon}\right|_{s=s_1} = \left.\frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon}\right|_{s=s_2} = 0\,, \tag{48}$$

but thanks to the G-invariance of the metric the ϵ variable is cyclic, $\frac{\delta\mathcal{L}}{\delta\epsilon} = 0$, and then the equations, together with the boundary conditions $\frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon}\big|_{s=s_1} = \frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon}\big|_{s=s_2} = 0$, imply

$$\mathrm{d}\left(\frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon}\right) = \left.\frac{\delta\mathrm{d}\mathcal{L}}{\delta\epsilon}\right|_{s=s_1} = \left.\frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon}\right|_{s=s_2} = 0 \quad \Rightarrow \quad \frac{\delta\mathrm{d}\mathcal{L}}{\delta\mathrm{d}\epsilon} = 0\,. \tag{49}$$

What we have here found are the conditions of horizontality. This procedure doesn't work if the metric is not G-invariant because then the 'Lagrangian' depends on $\epsilon(s)$ as well as on $\mathrm{d}\epsilon(s)$.

Notice that the free-end-point variation of a cyclic coordinate is equivalent to the regular variation of a Lagrange multiplier: if we define $\lambda = \mathrm{d}\epsilon$, then the Lagrangian depends on λ but not on its derivative. Its fixed-end-point variation gives $\frac{\delta\mathrm{d}\mathcal{L}}{\delta\lambda} = 0$, which is equivalent to (49).[26]

[26] In gauge theory, there are many 'multipliers' (like the scalar potential A_0 in Maxwellian electrodynamics) that, dimensionally, are velocities (and hence cyclic coordinates because no quantities of which they are velocities appear in the Lagrangian). It is therefore strictly irregular to treat them as multipliers. Free-end-point variation explains why mathematics that strictly is incorrect gives the right answer.

Stage II: Noether's theorem part I

Now, if the metric is G-invariant, the quantities $\frac{\delta \mathrm{d}\mathcal{L}}{\delta \mathrm{d}\epsilon}$ are *constants of motion* for the paths that minimize the following uncorrected action:

$$S_{\text{bare}} = \int \mathrm{d}\mathcal{L}_{\text{bare}} = \int \|\mathrm{d}\mathbf{r}_a\|_{\mathrm{G}} \,. \tag{50}$$

In fact, the action is invariant under *global*, time-independent G transformations

$$\mathbf{r}_a(s) \to \exp \epsilon \, \mathbf{r}_a(s) \,, \tag{51}$$

and Noether's theorem, part I [39] establishes that

$$\mathrm{d} \left(\left. \frac{\delta \mathrm{d}\mathcal{L}_{\text{bare}}(\epsilon \, \mathbf{r}_a(s))}{\delta \epsilon} \right|_{\epsilon=0} \right) = 0 \,, \tag{52}$$

which states that the quantities $\frac{\delta \mathrm{d}\mathcal{L}}{\delta \mathrm{d}\epsilon}$ of the preceding paragraph are conserved along the solutions. This is true, in particular, if they have the value zero. *Therefore the solutions of the Euler–Lagrange equations for S_{bare} are horizontal if they start horizontal:* $\left.\frac{\delta \mathrm{d}\mathcal{L}}{\delta \mathrm{d}\epsilon}\right|_{s=s_1} = 0$.

Thanks to the G-invariance of our bare action, the best-matching condition reduces to nothing more than an *initial condition* on the data: it is sufficient to take the 'bare' action S_{bare} and find the path that minimizes it with initial conditions given by (49).

The Newtonian N-body problem

Let's apply the technique of best-matching to Newtonian gravity. I will use the reparametrization-irrelevant formulation (37) with the variations expressed with respect to ephemeris time. It is sufficient to take the following metric on Q^N:

$$\mathrm{d}\mathcal{L}^2_{\text{New}} = 4\,(E - V_{\text{New}}) \sum_{a=1}^{N} \frac{m_a}{2} \mathrm{d}\mathbf{r}_a \cdot \mathrm{d}\mathbf{r}_a \,, \tag{53}$$

which is conformally related to the kinetic metric through the positive-definite[27] conformal factor $E - V_{\text{New}}$, where

$$V_{\text{New}} = - \sum_{a<b} \frac{m_a \, m_b}{\|\mathbf{r}_a - \mathbf{r}_b\|} \,. \tag{54}$$

The conformal factor $E - V_{\text{New}}$ is translation- and rotation-invariant, which allows us to find the horizontal curves by minimizing the action

$$\int \mathrm{d}\mathcal{L}_{\text{New}} = 2 \int \left((E - V_{\text{New}}) \sum_{a=1}^{N} \frac{m_a}{2} \, \|\mathfrak{D}\mathbf{r}_a\|^2 \right)^{\frac{1}{2}} , \tag{55}$$

[27] $E - V_{\text{New}}$ is actually positive-definite 'on-shell', that is, on the physical trajectories, where it is equal to the positive-definite kinetic energy. Notice that if $E < 0$, there are forbidden regions in configuration space: those where the potential energy is smaller than E.

where the best-matching differential here is

$$\mathfrak{D}\mathbf{r}_a = \mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega} \times \mathbf{r}_a + \mathrm{d}\boldsymbol{\theta} \,. \tag{56}$$

Varying wrt $\mathrm{d}\boldsymbol{\omega}$ with free endpoints, I find

$$\frac{\delta \mathrm{d}\mathcal{L}_{\text{New}}}{\delta \mathrm{d}\boldsymbol{\omega}} = - \frac{1}{\mathrm{d}\chi} \sum_{a=1}^{N} m_a \, \mathfrak{D}\mathbf{r}_a \times \mathbf{r}_a = 0 \,, \tag{57}$$

where

$$\mathrm{d}\chi = (E - V_{\text{New}})^{-\frac{1}{2}} \Big(\sum_a \tfrac{m_a}{2} \|\mathfrak{D}\mathbf{r}_a\|^2\Big)^{\frac{1}{2}} \,. \tag{58}$$

The expression here on the rhs is a sort of precursor of the ephemeris time, but it is not yet that since it depends on the auxiliary quantities $\mathrm{d}\boldsymbol{\omega}$ and $\mathrm{d}\boldsymbol{\theta}$; I introduce $\mathrm{d}\chi$ merely to simplify the equations.[28] As I commented on Eq. (40), E. Anderson calls it the *differential of the instant* [35, 40].

Varying wrt $\mathrm{d}\boldsymbol{\theta}$, I get

$$\frac{\delta \mathrm{d}\mathcal{L}_{\text{New}}}{\delta \mathrm{d}\boldsymbol{\theta}} = \frac{1}{\mathrm{d}\chi} \sum_{a=1}^{N} m_a \, \mathfrak{D}\mathbf{r}_a = 0 \,. \tag{59}$$

If we now define the *canonical momenta* [41, 42] $\mathbf{p}^a$ as

$$\mathbf{p}^a = \frac{\delta \mathrm{d}\mathcal{L}_{\text{New}}}{\delta \mathrm{d}\mathbf{r}_a} = m_a \frac{\mathfrak{D}\mathbf{r}_a}{\mathrm{d}\chi} \,, \tag{60}$$

then the Euler–Lagrange equations read

$$\mathrm{d}\mathbf{p}^a = \frac{\delta \mathrm{d}\mathcal{L}_{\text{New}}}{\delta \mathbf{r}_a} = - \mathrm{d}\chi \frac{\partial V_{\text{New}}}{\partial \mathbf{r}_a} - \mathrm{d}\boldsymbol{\omega} \times \mathbf{p}^a \,, \tag{61}$$

where the term $\mathrm{d}\boldsymbol{\omega} \times \mathbf{p}^a$ is due to the $\mathbf{r}_a$-dependence of $\mathfrak{D}\mathbf{r}_a$. This term can be reabsorbed into a best-matched differential of the momentum:[29]

$$\mathfrak{D}\mathbf{p}^a = \mathrm{d}\mathbf{p}^a + \mathrm{d}\boldsymbol{\omega} \times \mathbf{p}^a \,, \tag{62}$$

and then the equations of motion take an almost-Newtonian form:

$$\frac{\mathfrak{D}\mathbf{p}^a}{\mathrm{d}\chi} = m_a \frac{\mathfrak{D}}{\mathrm{d}\chi} \left(\frac{\mathfrak{D}\mathbf{r}_a}{\mathrm{d}\chi} \right) = - \frac{\partial V_{\text{New}}}{\partial \mathbf{r}_a} \,. \tag{63}$$

[28] I use χ for this differential quantity as the initial letter of the Greek word χρόνος (chronos), referring to a *quantitative* notion of time (duration).

[29] The best-matched differential cannot act in the same way on $\mathbf{r}_a$ and the momenta: for example the latter are translation-invariant. This will be made more precise in the Hamiltonian formulation.

As I said, $\mathrm{d}\chi$ is not an ephemeris time because of its dependence on the auxiliary quantities $\mathrm{d}\boldsymbol{\omega}$ and $\mathrm{d}\boldsymbol{\theta}$. However, if we solve (59) and (57) for $\mathrm{d}\boldsymbol{\omega}$ and $\mathrm{d}\boldsymbol{\theta}$, call the solutions $\mathrm{d}\boldsymbol{\omega}_{\text{BM}}$ and $\mathrm{d}\boldsymbol{\theta}_{\text{BM}}$ and substitute them into (58), we do obtain the actual ephemeris time:

$$\mathrm{d}t_{\text{eph}} = (E - V_{\text{New}})^{-\frac{1}{2}} \left(\sum_a m_a \|\mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega}_{\text{BM}} \times \mathbf{r}_a + \mathrm{d}\boldsymbol{\theta}_{\text{BM}}\|^2 \right)^{\frac{1}{2}} . \tag{64}$$

We can call $\mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega}_{\text{BM}} \times \mathbf{r}_a + \mathrm{d}\boldsymbol{\theta}_{\text{BM}}$ the 'horizontal' differential because it measures the physical variation of $\mathbf{r}_a$ and contains no part due to a rigid translation or rotation of the whole universe, and is therefore invariant under Eucl. The ephemeris time is invariant as well, and it measures a distillation of all the *physical* change in the universe.

The best-matching conditions (57), (59) take a very intuitive form if expressed in terms of the canonical momenta:

$$\mathbf{L} = \sum_{a=1}^N \mathbf{r}_a \times \mathbf{p}^a = 0 \,, \qquad \mathbf{P} = \sum_{a=1}^N \mathbf{p}^a = 0 \,. \tag{65}$$

We can now exploit the G-invariance of the metric as a shortcut to stage II of the best-matching procedure. The paths that minimize the bare action

$$S_{\text{bare}} = 2 \int \left((E - V_{\text{New}}) \sum_{a=1}^N \frac{m_a}{2} \|\mathrm{d}\mathbf{r}_a\|^2 \right)^{\frac{1}{2}} \tag{66}$$

and start with zero total angular and linear momenta,

$$\sum_{a=1}^N m_a \, \mathrm{d}\mathbf{r}_a = \sum_{a=1}^N m_a \, \mathrm{d}\mathbf{r}_a \times \mathbf{r}_a = 0 \,, \tag{67}$$

project under $\mathsf{Q}^N \to \mathsf{Q}^N_R$ to the physical solutions of the best-matched theory in Q^N_R. Minimizing S_{bare} with the above initial conditions gives Euler–Lagrange equations (38) that take a Newtonian form if expressed in terms of the ephemeris time for which $T_{\text{kin}} = E - V_{\text{New}}$.

If the energy is zero, $E = 0$, we see here how this kind of dynamics satisfies the Mach–Poincaré principle. The physical, observable initial data ($3N - 6$ positions and $3N - 6$ differentials) are alone enough to uniquely determine a physical trajectory. The total angular and linear momenta are set to zero by the constraints (65) (the angular momentum represents the 'missing' 3 data in Poincaré's analysis). The dynamics determines by itself an inertial frame of reference and an intrinsic notion of duration (the ephemeris time) such that Newton's equations hold. In this frame, the total angular momentum of the Universe must be zero, otherwise the Mach–Poincaré principle would be violated. We see that relational dynamics not only provides a deeper and intrinsic foundation for Newton's dynamics (finding the physical origin of reference frames and, ultimately, inertia), but it also imposes physical predictions that make it *more restrictive* than Newton's theory. If we could show that our Universe possesses angular momentum, that would rule out Machian dynamics (linear momentum wouldn't be observable anyway due to Galilean invariance).

A nonzero energy is an element of arbitrariness that would imply a (mild) violation of the Mach–Poincaré principle: observable initial data would fail to be enough to determine the future evolution, by just a single datum, and an observation of one single second derivative of the relational data would be enough to fix the value of E. Therefore a nonzero energy, despite being possible to describe in Machian terms, is disfavoured.

The scale-invariant N-body problem

If we want to incorporate the relativity of scale in a simple way, we might choose a different metric on Q^N which is manifestly scale-invariant:

$$\mathrm{d}\mathcal{L}^2_{\text{S}} = -4\, V_{\text{S}} \sum_{a=1}^N \frac{m_a}{2} \mathrm{d}\mathbf{r}_a \cdot \mathrm{d}\mathbf{r}_a \,, \qquad V_{\text{S}} = \frac{V_{\text{New}}}{I_{\text{cm}}^{\frac{1}{2}}} \,, \tag{68}$$

where I_{cm} is what I call the 'centre-of-mass moment of inertia':[30]

$$I_{\text{cm}} = \sum_{a=1}^N m_a \|\mathbf{r}_a - \mathbf{r}_{\text{cm}}\|^2 \,, \tag{69}$$

or, written in a more relational way (due to Leibniz),

$$I_{\text{cm}} = \sum_{a<b} \frac{m_a \, m_b}{m_{\text{tot}}} \|\mathbf{r}_a - \mathbf{r}_b\|^2 \,. \tag{70}$$

Notice that I can't put values of the energy E other than zero in (68) because that would make the metric non-scale invariant.

The best-matching action is

$$\int \mathrm{d}\mathcal{L}_{\text{S}} = 2 \int \left(- V_{\text{S}} \sum_{a=1}^N \frac{m_a}{2} \|\mathfrak{D}\mathbf{r}_a\|^2 \right)^{\frac{1}{2}} , \tag{71}$$

where in this case

$$\mathfrak{D}\mathbf{r}_a = \mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega} \times \mathbf{r}_a + \mathrm{d}\phi \, \mathbf{r}_a + \mathrm{d}\boldsymbol{\theta} \,. \tag{72}$$

[30] That object is actually half the trace of the centre-of-mass inertia tensor, defined as $\mathrm{I} = \sum_a m_a \, (\mathbf{r}_a - \mathbf{r}_{\text{cm}}) \otimes (\mathbf{r}_a - \mathbf{r}_{\text{cm}})$.

Note the scalar auxiliary variable $\phi \in \mathbb{R}^+$, which corrects for dilatations. The free-end-point variations of this action are identical in form to the ones of the previous paragraph. These now include the one related to scale transformations:

$$\frac{\delta \mathrm{d}\mathcal{L}_{\mathrm{S}}}{\delta \mathrm{d}\phi} = \frac{1}{\mathrm{d}\chi} \sum_{a=1}^{N} m_a \, \mathbf{r}_a \cdot (\mathfrak{D}\mathbf{r}_a) = 0 \,, \tag{73}$$

which expresses the vanishing of the total *dilatational momentum*:[31]

$$\mathrm{D} = \sum_{a=1}^{N} \mathbf{r}_a \cdot \mathbf{p}^a = 0 \,. \tag{74}$$

The equations of motion are

$$\frac{\mathfrak{D}\mathbf{p}^a}{\mathrm{d}\chi} = -\frac{\partial V_{\mathrm{S}}}{\partial \mathbf{r}_a} \,, \tag{75}$$

where in this case the best-matching differential of the momentum gains a correction coming from dilatations:

$$\mathfrak{D}\mathbf{p}^a = \mathrm{d}\mathbf{p}^a + \mathrm{d}\boldsymbol{\omega} \times \mathbf{p}^a - \mathrm{d}\phi \, \mathbf{p}^a \,. \tag{76}$$

In this case, the bare action,

$$S_{\mathrm{bare}} = 2 \int \left(- V_{\mathrm{S}} \sum_{a=1}^{N} \frac{m_a}{2} \, \|\mathrm{d}\mathbf{r}_a\|^2 \right)^{\frac{1}{2}} \,, \tag{77}$$

also conserves the dilatational momentum, and therefore its Euler–Lagrange equations give a representation in Q^N of the physical trajectories if one imposes the condition (74) on the initial data. The equations of motion for the bare action in ephemeris time are

$$\frac{\mathrm{d}\mathbf{p}^a}{\mathrm{d}t_{\mathrm{eph}}} = m_a \frac{\mathrm{d}^2\mathbf{r}_a}{\mathrm{d}t_{\mathrm{eph}}^2} = -\frac{\partial V_{\mathrm{S}}}{\partial \mathbf{r}_a} = -I_{\mathrm{cm}}^{\frac{1}{2}} \frac{\partial V_{\mathrm{New}}}{\partial \mathbf{r}_a} + m_a \left(\mathbf{r}_a - \mathbf{r}_{\mathrm{cm}} \right) I_{\mathrm{cm}}^{-\frac{1}{2}} \, V_{\mathrm{New}} \,. \tag{78}$$

The dilatational momentum is zero, and this implies that the center-of-mass moment of inertia is conserved, because

$$\frac{\mathrm{d}I_{\mathrm{cm}}}{\mathrm{d}t_{\mathrm{eph}}} = 2 \, \mathrm{D} = 0 \,. \tag{79}$$

[31] This quantity has the same dimensions as angular momentum, and the name for it was coined by analogy in [43]. It has not been given a name in the N-body literature and is usually denoted by $\mathcal{J}$, probably for Jacobi.

We therefore obtain a theory which is Newtonian gravity with $I_{\mathrm{cm}}^{\frac{1}{2}}$ playing the role of a gravitational constant, plus a 'cosmic' force parallel to $\mathbf{r}_a - \mathbf{r}_{\mathrm{cm}}$, and therefore pointing towards (or away from) the center of mass, which keeps I_{cm} constant. In a universe made of at least $\sim 10^{80}$ particles, this 'cosmic' force would be virtually undetectable through local observations (at the scale of the solar system or even at the scale of galaxy clusters). In fact, the accelerations of localized systems due to this force would be almost identical, both in direction and magnitude, and thus undetectable by virtue of the equivalence principle.

Further reading: Regarding principal fibre bundles, I suggest the book by Göckler and Schucker [44], the review by Eguchi, Gilkey and Hanson [45], and Frankel's book [32]. The masterpieces on the variational principles of mechanics are Lanczos [41], Goldstein [42] and Arnold [46].

6 Hamiltonian formulation

The Hamiltonian formulation of the kind of systems we're interested in is non-trivial. In fact the standard formulation fails to be predictive, precisely because of the relational nature of our dynamics. There are redundancies in the description, and this means that the usual Legendre transform that is used to define the Hamiltonian is singular, and the momenta are related to the velocities by one-to-many mappings. This situation is described through *nonholonomic constraints* (meaning constraints that depend not only on the coordinates, but also on the momenta).

In fact, after the two-stage procedure described above, best-matching leads to a simple set of constraints on the canonical momenta, namely Eqs. (65), with the possible addition of (74) if relativity of scale is assumed. There is also a constraint associated to temporal relationalism, as I will show now.

6.1 The Hamiltonian constraint

Reparametrization-invariant theories are characterized by having a vanishing Hamiltonian. We can see it in the prototype for these theories: the geodesic-generating action (35), whose Lagrangian is

$$\mathcal{L} = \left(g^{ij} \frac{\mathrm{d}x_i}{\mathrm{d}s} \frac{\mathrm{d}x_j}{\mathrm{d}s} \right)^{\frac{1}{2}} \,. \tag{80}$$

Its canonical momenta are

$$p^i = \frac{\delta \mathcal{L}}{\delta \dot{x}_i} = \left(g^{k\ell} \frac{\mathrm{d}x_k}{\mathrm{d}s} \frac{\mathrm{d}x_\ell}{\mathrm{d}s} \right)^{-\frac{1}{2}} g^{ij} \frac{\mathrm{d}x_j}{\mathrm{d}s} \,, \tag{81}$$

We see that the momenta, like the geodesic action $\int \mathrm{d}s\, \mathcal{L}$ from which they are derived, are themselves reparametrization-invariant: they can be written in the parametrization-independent form

$$p^i = \left(g^{k\ell}\, \mathrm{d}x_k\, \mathrm{d}x_\ell\right)^{-\frac{1}{2}} g^{ij}\, \mathrm{d}x_j\,. \tag{82}$$

The other thing one can observe is that the momenta have the form of N-dimensional *direction cosines*, because as a vector they have unit length, so the following phase-space function vanishes *weakly*:[32]

$$H = g_{ij}\, p^i\, p^j - 1 \approx 0\,, \tag{83}$$

where g^{ij} is the inverse metric. We see from (83) that the velocities $\frac{\mathrm{d}x_i}{\mathrm{d}s}$ are under-determined by the momenta (their norm is not determined by the momenta, which have unit norm). This is what Dirac [2] calls a *primary constraint*: an algebraic relation satisfied by the momenta by virtue of their mere definition and not due to any variation.

By its definition, the canonical Hamiltonian vanishes,

$$\mathcal{H}_{\text{can}} := \sum_i p^i\, \frac{\mathrm{d}x_i}{\mathrm{d}s} - \mathcal{L} \equiv 0\,, \tag{84}$$

but, as discussed below, the presence of the primary constraint (83) implies that the true generator of the dynamics is not (84) alone. It is instead (84) plus a linear combination of the primary constraints, which in our case is just (83):

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{can}} + u\, \left(g_{ij}\, p^i\, p^j - 1\right)\,, \tag{85}$$

where $u = u(s)$ is an arbitrary time-dependent function (whose conjugate momentum does not appear in the total Hamiltonian and is assumed to vanish).

Dirac [1] developed a general theory of constrained Hamiltonian systems and presented it in his beautiful lectures [2]. I will now reformulate everything according to Dirac's theory. For the readers who are not familiar with the subject, I will start with a quick review of the technique.

6.2 A crash course in Dirac's constraint analysis

Consider a Hamiltonian system which is subject to a set of constraints like (65), (74) and (83). The constraints will be expressed through a set of phase-space functions $\phi_a = \phi_a(p, q)$. When all of these functions vanish the constraints are satisfied. This identifies implicitly a hypersurface in phase space (the *constraint surface*). With the notion of constraints comes that of *weak equivalences*: two phase-space functions f, g are weakly equivalent if their difference is a linear combination of the constraints, $f \approx g \quad \Leftrightarrow f - g = \sum_a u_a\, \phi_a$. A function that is equal to a linear combination of the constraints (which means it is zero on the constraint surface) will be called *weakly vanishing*, and an equation that holds only on the constraint surface will be called a *weak equation*, as opposed to *strong equations*, which hold everywhere in phase space.

Dirac [2] starts by noticing that, in the presence of constraints, Hamilton's equations do not follow from the minimization of the canonical action $\delta \int \mathrm{d}s\, (p^i\, \dot{q}_i - \mathcal{H}_{\text{can}})$. In fact, when taking its variation,

$$\delta \int \mathrm{d}s\, (p^i\, \dot{q}_i - \mathcal{H}_{\text{can}}) = \int \mathrm{d}s \left[\left(\dot{q}_i - \tfrac{\delta \mathcal{H}_{\text{can}}}{\delta p^i}\right) \delta p^i - \left(\dot{p}^i + \tfrac{\delta \mathcal{H}_{\text{can}}}{\delta q_i}\right) \delta q_i\right] = 0\,, \tag{86}$$

one is not entitled to separately put to zero all the coefficients of δp^i and δq_i. This because one cannot take arbitrary variations δp^i, δq_i: they are constrained by the conditions $\phi_a \approx 0$. The most generic variation one can take is one that keeps the phase-space vector $(\delta p^i, \delta q_i)$ tangent to the hypersurface $\phi_a = 0$ (a variation that keeps you on that hypersurface).

There is no metric on phase space, but there is enough to define orthogonality and parallelism: it's the *symplectic structure*. The reader can find more details in Arnold's book [46]. For our purposes, it is sufficient to say that, in the case of a variation constrained to $\phi_a = 0$, Eq. (86) imposes a weaker set of conditions, namely

$$\frac{\delta \mathcal{H}_{\text{can}}}{\delta q_i} + \dot{p}^i = \sum_a u^a\, \frac{\partial \phi_a}{\partial q_i}\,, \qquad \frac{\delta \mathcal{H}_{\text{can}}}{\delta p_i} - \dot{q}_i = \sum_a u^a\, \frac{\partial \phi_a}{\partial p^i}\,, \tag{87}$$

for any choice of u_a. The u^a's purpose is to generate the whole tangent hyperplane to the surface $\phi_a = 0$ at each point of it. The above equations are Hamilton's equation for a *generalized Hamiltonian*

$$\mathcal{H}^* = \mathcal{H}_{\text{can}} + \sum_a u^a\, \phi_a\,. \tag{88}$$

The evolution of a phase-space function f under this generalized Hamiltonian can be written in terms of Poisson brackets

$$\dot{f} = \{f, \mathcal{H}^*\} = \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}^*}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial \mathcal{H}^*}{\partial q_i}\,. \tag{89}$$

The equations of motion (89) only make sense if the constraints ϕ_a are preserved by them, for otherwise the evolution brings us out of the constraint surface $\phi_a = 0$. So, for consistency, we have to require that

$$\dot{\phi}_a = \{\phi_a, \mathcal{H}^*\} = \{\phi_a, \mathcal{H}_{\text{can}}\} + \sum_b u^b \{\phi_a, \phi_b\} \approx 0\,. \tag{90}$$

[32] Strong ($=$) and weak ($\approx$) equations will be defined below. Here, the difference is immaterial.

Recall that Dirac's weak equality '≈ 0' means that the result of the above calculation gives a combination of the constraints ϕ_a that vanishes when the on-shell condition $\phi_a \approx 0$ is imposed.

Equations (90) are the core of Dirac's analysis. There are 4 cases:

1. There are a's for which Eqs. (90) have no solution. Then the system is not consistent and the equations of motion admit no sensible solution, like the famous Lagrangian $\mathcal{L} = q$, whose Euler–Lagrange equation is $1 = 0$.

2. Some of Eqs. (90) admit a nontrivial solution, and that solution does not depend on any u_a's. Then all such solutions impose new constraints ϕ'_a on the p^i, q_j. Dirac calls the ϕ'_a *secondary* constraints, and they need to be treated on the same footing as the ϕ_a's. Then one writes down a new modified Hamiltonian $\mathcal{H}^{**} = \mathcal{H}_{\text{can}} + \sum_a u^a \phi_a + \sum_b u'^b \phi'_b$ and applies the procedure again from the start.

3. Some other equations might admit a nontrivial solution which depends on the u^a's. Each such equation will fix one of the u^a's as a function of p, q. Dirac calls these equations 'specifiers'. Each equation of this kind is associated with a *second-class* constraint. Second-class constraints are defined by the fact that their Poisson brackets with at least one other constraint in the theory are not weakly vanishing.

4. All the equations that do not fall into cases 2 or 3 will simplify to tautologies of the form $1 = 1$. There will be one such equation for each *first-class* constraint. First-class constraint are defined by the fact that their Poisson brackets with every other constraint in the theory are weakly vanishing.

If we never stumble upon case 1, after a few iterations of the procedure all of the equations will fall into cases 3 or 4 Then the system is well defined, and we can stop. The evolution will be finally generated by a *total* Hamiltonian,

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{can}} + \sum_{a \,\in\, \textit{first-class}} u^a \phi_a + \sum_{b \,\in\, \textit{second-class}} u^b(p,q)\, \phi_b \,, \tag{91}$$

which is the canonical Hamiltonian plus a linear combination of all the leftover first-class constraints (primary, secondary, ...), and a linear combination of the second-class constraints with the $u^a = u^a(p,q)$ which have been fixed by the specifier equations.

Dirac's theorem

Whenever our algorithm stops (and the system turns out to be consistent), but some of the u^a's are not specified by any 'case 3' condition, then these u^a's are *gauge* degrees of freedom (like the position of the center of mass in relational systems).

As we said, the u^a's that don't end up specified are those related to *first-class* constraints, meaning that they Poisson-commute with all the other constraints ϕ_a and with $\mathcal{H}_{\text{can}}$, so that for them Eq. (90) falls into case 4.

The following is referred to by some as 'Dirac's theorem': *primary first-class constraints treated as generating functions of infinitesimal contact transformations lead to changes that do not affect the physical state.*

Consider the simplest case of a non-vanishing canonical Hamiltonian $\mathcal{H}_{\text{can}}$ and one single first-class constraint ϕ:

$$\dot{f} = \{f, \mathcal{H}_{\text{can}}\} + u\,\{f, \phi\}.$$

If ϕ is first-class (so that $\{\phi, \mathcal{H}_{\text{can}}\} \approx 0$), then u is not specified by any condition and is left as arbitrary. This arbitrariness is absent only if one considers phase-space variables y that are first-class with respect to ϕ, $\{y, \phi\} \approx 0$. Then u does not appear in the evolution of y. Variables like y are gauge-invariant, and the absence of any arbitrariness in their evolution signals that they are physical and are the only candidates for observables. Non-gauge invariant quantities can be used in the description of the system, but they depend on conventional choices, or gauges.

Let's show a simple example to make things concrete. Consider two free particles on a line, with coordinates q_1 and q_2. The canonical Hamiltonian is

$$\mathcal{H}_{\text{can}} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \,, \tag{92}$$

and say we have the constraint

$$\phi = p_1 + p_2 \,, \tag{93}$$

which is obviously first-class as it is the one and only constraint in the model, and a single constraint will always Poisson-commute with itself. The total Hamiltonian is $\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{can}} + u\,\phi$, and it generates the time evolution

$$\dot{q}_1 = \frac{p_1}{m_1} + u \,, \qquad \dot{q}_2 = \frac{p_2}{m_2} + u \,, \,, \qquad \dot{p}_1 = \dot{p}_2 = 0 \,. \tag{94}$$

The solution to these equations is

$$q_i(s) = \frac{p_i}{m_i}\,(s - s_1) + \int_{s_1}^{s} \mathrm{d}s'\, u(s') + q_i(s_1) \,, \qquad p_i(s) = p_i(s_1) \,, \tag{95}$$

and depends on four integration constants, $q_i(s_1)$, $p_i(s_1)$, and the arbitrary parameter $u(s)$. The integration constants set the initial values of the phase-space variables, at $s = s_1$. The constraint $\phi \approx 0$ constrains these initial values to satisfy $p_1(s_1) = -p_2(s_1)$.

We see that while the constraint $\phi \approx 0$ fixed the total canonical momentum $p_1 + p_2$ to be zero, the gauge choice allows us to represent the system in a frame in which the total quantity $m_1 \dot{q}_1 + m_2 \dot{q}_2 = m_{\text{tot}} u$ is nonzero. The relative distance between the two particles $q_1 - q_2$ is gauge-invariant $\{q_1 - q_2, \phi\} = 0$, its equations of motion contain no arbitrariness,

$$\dot{q}_1 - \dot{q}_2 = \frac{p_1}{m_1} - \frac{p_2}{m_2}\,, \tag{96}$$

and it is the only gauge-invariant quantity in the system apart from the momenta p_1, p_2 (which however are constrained to sum to zero, and therefore have only one independent degree of freedom). Therefore, there are two physical Hamiltonian degrees of freedom.

Barbour and Foster's exception to Dirac's theorem

Julian Barbour and Brendan Foster found a bug in Dirac's theorem [3] that relates to the case we're interested in: when the canonical Hamiltonian vanishes and the total Hamiltonian is just a linear combination of constraints.

Consider again the case of a single first-class constraint ϕ, but when the canonical Hamiltonian vanishes identically:[33] $\mathcal{H}_{\text{can}} \equiv 0$. Then

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}s} = u\,\{f, \phi\}\,, \tag{97}$$

and now it is true that u is arbitrary, but the effect of changing u is just a reparametrization. The same equation written as

$$\frac{1}{u}\frac{\mathrm{d}f}{\mathrm{d}s} = \{f, \phi\}$$

is invariant under $s \to s'(s)$, $u \to u\,\frac{\partial s}{\partial s'}$. Therefore this particular ϕ generates physical change. If in the example above of the two particles on a line we had $\mathcal{H}_{\text{can}} = 0$, then the equations of motion would reduce to

$$\dot{q}_1 = u\,, \qquad \dot{q}_2 = u\,,\,, \qquad \dot{p}_1 = \dot{p}_2 = 0\,. \tag{98}$$

with solution

$$q_i = \int_{s_1}^{s} \mathrm{d}s'\, u(s') + q_i(s_1)\,, \qquad p_i = const. \tag{99}$$

Now, we cannot say that the constraint generates unphysical change, because it clearly moves the representative point on a dynamical trajectory. Changing the function $u = u(s)$ merely amounts to changing the parametrization of the same dynamical trajectory.

[33] 'Strongly' in Dirac's terminology.

6.3 Application to our systems

6.3.1 Differential almost-Hamiltonian formulation

We will here use the formulation proposed by Edward Anderson [35] as in Eq. (37). In place of the Lagrange multipliers u^a, we use differentials $\mathrm{d}\xi^a$ of which we consider the free-end-point variation.[34] This enables us to implement 'parametrization irrelevance' at all stages. The Newtonian best-matching action (55) is extremalized by a path generated by the 'differential-almost-Hamiltonian' object

$$\mathrm{d}\mathcal{A} = \mathrm{d}\chi\,\mathrm{H} + \mathrm{d}\boldsymbol{\theta}\cdot\mathbf{P} + \mathrm{d}\boldsymbol{\omega}\cdot\mathbf{L}\,, \tag{100}$$

which is a linear combination (through differentials) of the constraints

$$\mathbf{L} = \sum_{a=1}^{N} \mathbf{r}_a \times \mathbf{p}^a\,, \qquad \mathbf{P} = \sum_{a=1}^{N} \mathbf{p}^a\,, \qquad \mathrm{H} = \sum_{a=1}^{N} \frac{\mathbf{p}^a\cdot\mathbf{p}^a}{2\,m_a} - U\,. \tag{101}$$

The evolution of a phase-space function f is generated by Poisson-commuting f with $\mathrm{d}\mathcal{A}$:

$$\mathrm{d}f = \{\mathrm{d}\mathcal{A}, f\} = \mathrm{d}\chi\,\{\mathrm{H}, f\} + \mathrm{d}\boldsymbol{\theta}\cdot\{\mathbf{P}, f\} + \mathrm{d}\boldsymbol{\omega}\cdot\{\mathbf{L},, f\}\,, \tag{102}$$

and therefore the best-matching differential of f is generated by the Hamiltonian constraint H 'smeared' with $\mathrm{d}\chi$,[35]

$$\mathfrak{D}f = \mathrm{d}\chi\,\{\mathrm{H}, f\} = \mathrm{d}f - \mathrm{d}\boldsymbol{\theta}\cdot\{\mathbf{P}, f\} - \mathrm{d}\boldsymbol{\omega}\cdot\{\mathbf{L}, f\}\,. \tag{103}$$

One sees immediately that this definition reproduces the correct action of the best-matching differential on the coordinates and the momenta:

$$\begin{aligned} \mathrm{d}\mathbf{q}_a - \{\mathrm{d}\boldsymbol{\theta}\cdot\mathbf{P}, \mathbf{q}_a\} - \{\mathrm{d}\boldsymbol{\omega}\cdot\mathbf{L}, \mathbf{q}_a\} &= \mathrm{d}\mathbf{q}_a + \mathrm{d}\boldsymbol{\theta} + \mathrm{d}\boldsymbol{\omega}\times\mathbf{q}_a\,, \\ \mathrm{d}\mathbf{p}^a - \{\mathrm{d}\boldsymbol{\theta}\cdot\mathbf{P}, \mathbf{p}^a\} - \{\mathrm{d}\boldsymbol{\omega}\cdot\mathbf{L}, \mathbf{p}^a\} &= \mathrm{d}\mathbf{p}^a + \mathrm{d}\boldsymbol{\omega}\times\mathbf{p}^a\,. \end{aligned} \tag{104}$$

The equations of motion are

$$\mathfrak{D}\mathbf{r}_a = \mathrm{d}\chi\,\{\mathrm{H}, \mathbf{r}_a\} = \mathrm{d}\chi\,\frac{\mathbf{p}^a}{m_a}\,, \qquad \mathfrak{D}\mathbf{p}^a = \mathrm{d}\chi\,\{\mathrm{H}, \mathbf{p}^a\} = \mathrm{d}\chi\,\frac{\partial U}{\partial \mathbf{r}_a}\,. \tag{105}$$

The angular and linear momentum constraints close as a first-class system:

$$\{\mathrm{L}_i, \mathrm{P}_j\} = \epsilon_{ijk}\,\mathrm{P}_k\,, \quad \{\mathrm{P}_i, \mathrm{P}_j\} = 0\,, \tag{106}$$

[34] As we saw in Sec. 5.2 the free-end-point variation of a cyclic coordinate is equivalent to the regular variation of a Lagrange multiplier.

[35] 'Smearings' are defined below in the Relational Field Theory Part.

but their Poisson brackets with the Hamiltonian constraint depend on the potential U:

$$\{\mathbf{L}, \mathrm{H}\} = \sum_{a=1}^N \mathbf{r}_a \times \frac{\partial U}{\partial \mathbf{r}_a} \,, \quad \{\mathbf{P}, \mathrm{H}\} = \sum_{a=1}^N \frac{\partial U}{\partial \mathbf{r}_a} \,. \tag{107}$$

The only way to make the constraints propagate and obtain a consistent theory is to have a potential that is invariant under global translations and rotations, so that both of the above commutators vanish strongly.

If we insist on the relativity of scale, we have to add the dilatational momentum constraint

$$\mathrm{D} = \sum_{a=1}^N \mathbf{r}_a \cdot \mathbf{p}^a \,, \tag{108}$$

which is first-class with respect to the momentum constraints, but the same holds also for the Hamiltonian constraint only for some choices of the potential,

$$\{L_i, \mathrm{D}\} = 0 \,, \quad \{\mathrm{D}, P_i\} = P_i \,, \quad \{\mathrm{H}, \mathrm{D}\} = - \sum_{a=1}^N \frac{\mathbf{p}^a \cdot \mathbf{p}^a}{m_a} + \sum_{a=1}^N \mathbf{r}_a \cdot \frac{\partial U}{\partial \mathbf{r}_a} \,, \tag{109}$$

namely, the energy has to be zero $E = 0$ and the potential has to be *homogeneous of degree -2* in order for $\{\mathrm{H}, \mathrm{D}\}$ to weakly vanish. This is a consequence of Euler's homogeneous function theorem [47].

6.4 A matter of units

Before moving to field theory, I want to make some remarks about dimensional analysis, which in a relational setting becomes a key – and nontrivial – point. There is much confusion about the role of units in physics,[36] and the relational point of view highlights the issue and calls for clear thinking. I'll set the stage for my argument here, using only relational particle dynamics as my prototype Machian theory. The starting point is the best-matching action for Newtonian gravity (55). The particle coordinates will be assumed of course to carry the dimensions of a length $[\mathbf{r}_a] = \ell$. The masses are usually given dimension, but they are non-dynamical objects, as they are constant in time and do not evolve. They are therefore 'transparent' under all derivatives, and one can always remove the mass dimension from any equation. This can be easily achieved by dividing the action by the 3/2th power of a reference mass, which is naturally assumed to be the total mass $m_{\text{tot}} = \sum_{a=1}^N m_a$. Then the action only depends on dimensionless 'geometric' masses $\mu_a = m_a/m_{\text{tot}}$ (notice how I rescaled and changed the units of the energy $E' = E/m_{\text{tot}}^2$ which I can do because it is just a constant):

$$\int \mathrm{d}\mathcal{L}_{\text{New}} = 2 \int \left((E' - V_{\text{New}}/m_{\text{tot}}^2) \sum_{a=1}^N \frac{\mu_a}{2} \|\mathfrak{D}\mathbf{r}_a\|^2 \right)^{\frac{1}{2}} \,.$$

Let's now talk about Newton's constant G: in our expression for Newton's potential (54) it didn't feature. In basic physics courses G is introduced as a conversion factor from mass$\times$length^{-1} to accelerations. An acceleration is length$\times$time^{-2}, but in our framework the independent variable is dimensionless: it is just a parameter on the evolution curve in configuration space, $[s] = 1$. It is ephemeris time dt_{eph} (64) that plays the role of Newtonian absolute time. Its dimensions can be read off its expression (in the mass-rescaled case):

$$\mathrm{dt}_{\text{eph}} = (E' - V_{\text{New}}/m_{\text{tot}}^2)^{-\frac{1}{2}} \left(\sum_a \mu_a \|\mathrm{d}\mathbf{r}_a + \mathrm{d}\boldsymbol{\omega}_{\text{BM}} \times \mathbf{r}_a + \mathrm{d}\boldsymbol{\theta}_{\text{BM}}\|^2 \right)^{\frac{1}{2}} \,.$$

It is a length$^{3/2}$, $[\mathrm{dt}_{\text{eph}}] = \ell^{3/2}$. Newton's law follows from our use of t_{eph} for parametrization and of the best-matched coordinates $\mathbf{r}_a^{\text{BM}} = \mathbf{r}_a + \boldsymbol{\omega}_{\text{BM}} \times \mathbf{r}_a + \boldsymbol{\theta}_{\text{BM}}$,

$$\mu_a \frac{\mathrm{d}^2 \boldsymbol{r}_a^{\text{BM}}}{\mathrm{dt}_{\text{eph}}^2} = \sum_{a<b} \frac{\mu_a \mu_b}{\|\mathbf{r}_a^{\text{BM}} - \mathbf{r}_b^{\text{BM}}\|^3} (\mathbf{r}_a^{\text{BM}} - \mathbf{r}_b^{\text{BM}}) \,, \tag{110}$$

(notice the appearance of rescaled masses only). The above equation is dimensionally consistent without the need for any conversion factor – G simply doesn't appear. But what we call Newton constant and denote with G was in effect measured by Henry Cavendish a bit over 200 years ago, so what is it?

Let's consider its definition [49]: it is the gravitational force in Newtons exerted by one mass of 1 kg on another mass of 1 kg placed at a distance of 1 m. This definition is of course very unsatisfactory from a relational point of view, and we would like to express it as a comparison. But there is another quantity in particle mechanics which has precisely the same definition, only with electric charges in place of masses: it is Coulomb's constant. Its definition is [50]: the electrostatic force in Newtons exerted by one charge of 1 C on another charge of 1 C placed at a distance of 1 m. This suggests a different understanding of Newton's constant: it is just the (dimensionless!) relative magnitude between gravitational and other kind of interactions. If I were to include electrostatic interactions in my relational particle model, I would use a potential of this form:

$$V_{\text{N+C}} = - \sum_{a<b} \frac{\mu_a \mu_b}{\|\mathbf{r}_a - \mathbf{r}_b\|} - \sum_{a<b} \frac{\epsilon_a \epsilon_b}{\|\mathbf{r}_a - \mathbf{r}_b\|} \,, \tag{111}$$

[36] The reader might enjoy reading the interesting and inconclusive 'trialogue' between Duff, Okun and Veneziano [48] on the number of fundamental constants in nature.

where ϵ_a are dimensionless electric charges. If the particles we are considering are subatomic like the electron or the proton, the ϵ_a's are much larger than the μ_a's. In fact, for a given particle, $\epsilon_a/\mu_a = (k_e/G)^{\frac{1}{2}} e_a/m_a$ where now k_e is Coulomb's constant and G is Newton's constant if e_a is expressed in Coulombs and m_a in kg. There's no such a thing as a universal gravitation constant or a permittivity of vacuum: there are only (smaller or larger) dimensionless coupling constants. The story changes in presence of 'non-$1/r^2$' kinds of forces, like harmonic oscillators or the Lennard–Jones potential. If they are introduced in a naïve way, *e.g.*:

$$V_{\text{N+C+L-J}} = -\sum_{a<b} \frac{\mu_a \mu_b + \epsilon_a \epsilon_b}{\|\mathbf{r}_a - \mathbf{r}_b\|} - \epsilon_0 \sum_{a<b} \left(\frac{r_m^{12}}{\|\mathbf{r}_a - \mathbf{r}_b\|^{12}} - \frac{2\, r_m^6}{\|\mathbf{r}_a - \mathbf{r}_b\|^6} \right) , \qquad (112)$$

those forces clearly require the introduction of truly dimensionful constants, as the length r_m of Eq. (112) where the Lennard-Jones potential has its minumum, and ϵ_0 with dimensions ℓ^{-1}. A dimensionful constant like r_m represents a conceptual challenge: it gives an absolute scale to the Universe, which doesn't make reference to any dynamical quantity in it. In other words, a dimensionful constant is associated to a length, but what is it the length of?

As a first argument, we have to ask: how do we measure things like r_m or ϵ_0? The answer is straightforward: we measure the equilibrium distance between two atoms which interact through the Lennard-Jones potential. But that distance is always measured in relation to something else: all measurements are, by their own nature, relational. If we doubled all the distances in the universe there should be no visible effect: also r_m and ϵ_0^{-1} should be doubled. Therefore we should rather express r_m and ϵ_0^{-1} as dimensionless constants times some other distance in the universe, but which one? The problem is that, according to laboratory experience, r_m and ϵ_0 are, to a very good approximation, constant in time, while there are no distances in the universe which are truly constant. If we want to give an answer that satisfies the experimentalist, we can limit to a distance which is only approximately constant, within the experimental error: the natural choice is $r_m, \epsilon_0^{-1} \propto \sqrt{I_{\text{cm}}}$, the square root of the center-of-mass moment of inertia of the whole universe defined in (70), essentially measuring the size of the universe. As we remarked in section 5.2 the time dependence given by $r_m, \epsilon_0^{-1} \propto \sqrt{I_{\text{cm}}}$ would be very hard to detect in the lab, within human time frames.

I will now spell out a further argument in favour of $r_m, \epsilon_0^{-1} \propto \sqrt{I_{\text{cm}}}$: Van der Waals or harmonic forces can be considered as effective descriptions of more fundamental physics which has just $1/r^2$-type forces, in which one ignores or 'coarse-grains' some degrees of freedom. Think about the electrical dipole force generated by a pair of oppositely-charged particles: it falls off like $1/r^3$, but it is generated by purely $1/r^2$-type forces. Therefore the dimensionful constant appearing in the dipole potential must be related to some physical lengths: it is easy to convince oneself that it is the size of the orbits of the pair of charges. This means that if every size in the universe was scaled, the dimensionful constants would scale accordingly. Positing $r_m, \epsilon_0^{-1} \propto \sqrt{I_{\text{cm}}}$ finds then a justification in terms of effettive physics (or at least it appears more physical than having non-relational constants).

The two arguments above might seem convincing, but they ignore quantum mechanics. For instance, the argument about dipole forces doesn't make fully sense at a purely classical level: if we ignore quantum mechanics, the orbits of different pairs will have a continuum of different orbital element, and these orbital elements will change continuously due to interactions with the rest of the universe. It would be unrealistic to assume something like $r_m, \epsilon_0^{-1} \propto \sqrt{I_{\text{cm}}}$ where the proportionality factors are (even approximately) constant in time: in a classical world, these proportionality factors would change rapidly in time. Quantum mechanics changes the picture: the orbitals of electrons in an atom are quantized, and they can only jump by discrete quantities. At the deepest level, it is this discretization that allow us to talk about 'atoms', and to attribute to the dimensionless proportionality factors in $r_m, \epsilon_0^{-1} \propto \sqrt{I_{\text{cm}}}$ a set of unchanging discrete values. The issue of the origin of physical scales and units is a deep one, and requires a more careful discussion, which would go beyond the purposes of this Tutorial.

Further reading: For constrained systems I suggest Dirac's 'Lectures' [2], Henneaux-Teitelboim's book [51], and the book by Regge, Teitelboim [52]. Barbour and Foster's paper on Dirac's theorem [3], E. Anderson's review [35] for the 'differential-almost-Hamiltonian' approach.

Part III

Relational Field Theory

The ontology of fields

Faraday is credited with the introduction of the concept of field in physics. He found it extremely useful, in particular for the description of magnetic phenomena, to use the concept of *lines of force* (1830s) [53]:

> *3071. A line of magnetic force may be defined as that line which is described by a very small magnetic needle, when it is so moved in either direction correspondent to its length, that the needle is constantly a tangent to the line of motion; [...]*
>
> *3072. [...] they represent a determinate and unchanging amount of force. [...] the sum of power contained in any one section of a given portion of the lines is exactly equal to the sum of power in any other section of the same lines, however altered in form, or however convergent or divergent they may be at the second place. [...]*
>
> *3073.These lines have not merely a determinate direction, [...] but because they are related to polar or antithetical power, have opposite qualities in opposite directions; these qualities [...] are manifest to us, [...] by the position of the ends of the magnetic needle, [...]*

He speculated that the concept might be useful beyond magnetic phenomena,

> *3243. [...] The definition then given had no reference to the physical nature of the force at the place of action, and will apply with equal accuracy whatever that may be; [...]*

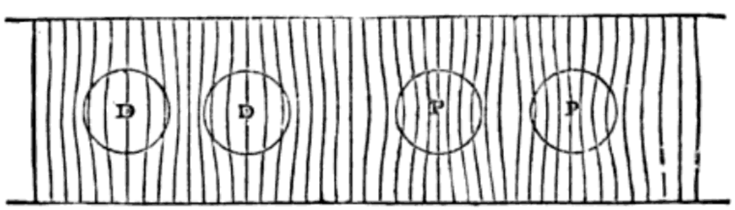

Figure 15: An illustration from Faraday's book [53] showing his lines of magnetic force through diamagnetic (D) and paramagnetic (P) materials.

> *3251. Three great distinctions at least may be taken among these cases of the exertion of force at a distance; that of gravitation, where propagation of the force by physical lines through the intermediate space is not supposed to exist; that of radiation, where the propagation does exist, and where the propagating line or ray, once produced, has existence independent either of its source, or termination; and that of electricity, where the propagating process has intermediate existence, like a ray, but at the same time depends upon both extremities of the line of force [...] Magnetic action at a distance has to be compared with these. It may be unlike any of them; for who shall say we are aware of all the physical methods or forms under which force is communicated? It has been assumed, however, by some, to be a pure case of force at a distance, and so like that of gravity; whilst others have considered it as better represented by the idea of streams of power. The question at presence appears to be, whether the lines of magnetic force have or have not a physical existence; and if they have, whether such physical existence has a static or dynamic form [...].*

To Faraday it appeared clear that the issue of *retardation* was key to determine whether physical existence should be attributed to the lines of force of a certain interaction:

> *3246. There is one question in relation to gravity, which, if we could ascertain or touch it, would greatly enlighten us. It is, whether gravitation requires* time*. If it did, it would show undeniably that a physical agency existed in the course of the line of force. [...]*
>
> *3247. When we turn to radiation phænomena, then we obtain the highest proof, that though nothing ponderable passes, yet the lines of force have a physical existence independent, in a manner, of the body radiating, or the body receiving the rays. [...]*

It was Maxwell who proved, with a monumental work, the superiority of the concept of fields for the description of electric and magnetic phenomena. In his 1855 seminal paper *On Faraday's Lines of Force* [54] Maxwell modelled the field with an incompressible fluid whose velocity defined the field intensity and whose flux lines coincided with Faraday's lines of force. This analogy is particularly suited for forces that fall off as the square of the distance, like the electric and magnetic ones. Maxwell's [54] ends with the following lines:

> *By referring everything to the purely geometrical idea of the motion of an imaginary fluid, I hope to attain generality and precision, and to avoid the dangers arising from a premature theory professing to*

explain the cause of the phenomena. If the results of mere speculation which I have collected are found to be of any use to experimental philosophers, in arranging and interpreting their results, they will have served their purpose, and a mature theory, in which physical facts will be physically explained, will be formed by those who by interrogating Nature herself can obtain the only true solution of the questions which the mathematical theory suggests.

In his masterpiece, *A Dynamical Theory of the Electromagnetic Field* [55], Maxwell deduces the speed of light according to his model from the values of the electric permittivity and magnetic permeability of air measured by Weber and Kohlrausch. He then compares it to the direct measurement of the speed of light in air due to Fizeau and Foucault, finding good agreement:

The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws. [...] Hence electromagnetic science leads to exactly the same conclusions as optical science with respect to the direction of the disturbances which can be propagated through the field; both affirm the propagation of transverse vibrations, and both give the same velocity of propagation.

The origins of Geometrodynamics

Carl Friedrich Gauss in his 1827 *Disquisitiones generales circa superficies curvas (General investigations of curved surfaces)* [56] studied parametrized (coordinatized) 2d surfaces embedded in 3d Euclidean space. He was interested in those properties of the surface that are unaffected by a change of the way the surface is embedded in 3d space (as, for example, bending the surface without stretching it), or a change in the parametrization of the surface. One natural such invariant quantity is the length of a curve drawn along the surface. Another is the angle between a pair of curves drawn along the surface and meeting at a common point, or between tangent vectors at the same point of the surface. A third such quantity is the area of a piece of the surface. The study of these invariants of a surface led Gauss to introduce the predecessor of the modern notion of metric tensor. Among the important notions introduced by Gauss, there is the concept of *intrinsic*, or *Gaussian* curvature, for which the famous "Theorema egregium" holds [56]:

If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.

Thus, the theorem states that the curvature of a surface can be determined entirely by measuring angles and distances on the surface, it does not depend on how the surface might be embedded in 3-dimensional space or on the parametrization of the surface.

In 1854 Bernard Riemann had to give a habilitation lecture at the University of Göttingen and proposed to Gauss three topics. Gauss chose the one on the foundations of geometry [57]. The lecture revolutionized geometry, generalizing Gauss' results to any dimension, opening the possibility that the 3-dimensional space in which we live and do physics might not be Euclidean and could possess intrinsic curvature [57]. Riemann argued for an *empirical* foundation for geometry:

Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined; a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure-relations of space - the most important system for our present purpose being that which Euclid has laid down as a foundation. These matters of fact are - like all matters of fact - not necessary, but only of empirical certainty; they are hypotheses. We may therefore investigate their probability, which within the limits of observation is of course very great, and inquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small.

Riemann's central concept was that of a metric, which characterizes the intrinsic geometry of a manifold. The importance of Riemann's work was so outstanding that now we talk about *Riemannian geometry.*

Following that, around the end of the 19th century, Gregorio Ricci-Curbastro and Tullio Levi-Civita [58] established the modern notions of tensors, as differential objects which are independent of the coordinate system, laying the basis of modern differential geometry.

For our purposes, Einstein took the last step in this story by identifying spacetime with a 4-dimensional manifold with Lorentzian signature, whose curvature was related to the energy–momentum tensor of matter through Einstein's equations:

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = 8\,\pi\, T_{\mu\nu}\,, \tag{113}$$

in units where $G = c = 1$. How this came about is an extremely interesting story that is recounted in great details elsewhere. My aim in this tutorial is to do something different.

An exercise in counterfactual history

Einstein's discovery was strongly guided by the revolution brought about by Special Relativity, after which the unity of space and time into a continuum appeared to be an inevitable property of any future theory of physics. Had, as it might have happened, the Machian ideas been made precise earlier by someone like Poincaré, Barbour and Bertotti, the history of physics would probably have taken a different course. I will now engage in this exercise in *counterfactual history:* imagine all I have explained about spatial and temporal relationalism in the previous Part had been understood in the 19th century, and add all the insights on the ontology of fields and geometry provided by Faraday, Maxwell, Gauss, Riemann and Ricci–Levi-Civita. What could a relationalist physicist have done to extend Mach's ideas to field theory?

She/he would have started by assuming a 3-dimensional perspective, in which space is a 3d Riemannian manifold described by a metric tensor g_{ab}, on which other kind of fields live (scalar, vectors...). Then she/he would have looked for a variational principle producing a dynamics which depended on the fields and their first derivatives and with a Jacobi-type action. Any arbitrariness in the description, like the coordinate system used to coordinatize the manifold and write tensors in components, should be made redundant with best-matching, in analogy to the particle models.

In the following Sec. 7 and 9 I will show how much could have been achieved with this handful of first principles. One can deduce the Special Relativity principle, the invariance of the speed of light and the universality of the light-cone. Moreover the whole of General Relativity can be derived, as well as the gauge principle, both in its abelian (electromagnetism) and non-abelian (Yang–Mills theory) form. In addition, the same principles allow for two different additional kinds of relativity (Galilean Relativity and Carrollian Relativity, or Strong Gravity) which are relevant in particular regimes of GR. Finally, by requiring the Mach–Poincaré principle to be satisfied by those theories, one unambiguously obtains the theory which is the main subject of this Tutorial: Shape Dynamics.

Between Secs. 7 and 9 I have inserted a Section, number 8, on York's method for solving the initial-value problem in General Relativity, which is necessary to understand the following, where I show how implementing the Mach–Poincaré principle leads to Shape Dynamics.

The following sections will stress the fact that Shape Dynamics is logically independent of General Relativity. It is in fact unnecessary to even know anything about GR to understand these sections, and all its main features will be derived independently, from more fundamental first principles. Needless to say, readers who are familiar with GR, and in particular its Hamiltonian formulation due to Arnowitt–Deser–Misner will be able to appreciate the following sections even more. I devote the whole Appendix A to reviewing the ADM formulation of GR. Readers unfamiliar with it should read Appendix A either now, before the next Sections 7-9, or after them, in order to understand the connection with the spacetime picture. Also many readers who are familiar with ADM gravity might have never seen the Baierlein–Sharp–Wheeler action, which is the key link between ADM and the formulations of the next sections, so they might consider reading Appendix A now.

Nuggets of functional analysis

In this Part I will be working on a 3-dimensional Riemannian manifold Σ, which is therefore endowed with a Riemannian 3-metric g_{ab} and an associated integration measure $\int_\Sigma \mathrm{d}^3x\sqrt{g}$. Throughout this Part I will assume the manifold to be closed (compact without boundary). This will be relaxed in Sec. 12 of the next Part, and in Sec. A.4 of Appendix A .

The fundamental objects of our study will be fields, understood as different multiplets of functions on Σ. The prototype for our configuration spaces will be $\mathscr{L}^2(\Sigma)$, the space of square-integrable functions on our spatial manifold Σ. This is also the configuration space of scalar fields. More complicated fields like tensors will just be tensor products of $\mathscr{L}^2(\Sigma)$ (one for each component), with particular transformation laws under diffeomorphisms and under the basic operations of tensor calculus (if the reader needs an introduction to tensor calculus I suggest starting with Schutz [59], continue with Göckeler–Schücker [44], and ending up with Frankel [32]).

$\mathscr{L}^2(\Sigma)$ can be made into an inner product space by defining

$$(f \,|\, h) = \int \mathrm{d}^3x \, \sqrt{g} \, f(x) \, h(x) \,, \tag{114}$$

which satisfies all the axioms of an inner product. This inner product can be extended to dual tensors, e.g.,

$$(T^{ij} \,|\, S_{ij}) = \int \mathrm{d}^3x \, \sqrt{g} \, T^{ij}(x) S_{ij}(x) \,, \tag{115}$$

and to tensor densities, e.g.,

$$(T^{ij} \,|\, S_{ij}) = \int \mathrm{d}^3x \, g^{\frac{y}{2}} \, T^{ij}(x) S_{ij}(x) \qquad y = 1 - w - z \,, \tag{116}$$

if T^{ij} is a density of weight w and S_{ij} is of weight z.

The constraints in field theory are actually *one constraint per space point* and *can contain spatial derivatives of the fields.* The proper way to understand field theory constraints is through *smearings*: I can transform a constraint $\chi(x)$ into a real number by integrating it against a test function $(\chi \,|\, f)$. Then $(\chi \,|\, f)$

becomes a *linear functional* of $f(x)$, of which one can take variations. For example, if the metric g_{ab} is the field we want to take variations of, we have

$$\delta(\chi\,|\,f) = \int_\Sigma \mathrm{d}^3x \frac{\delta(\chi\,|\,f)}{\delta g_{ij}(x)} \delta g_{ij}(x)\,, \tag{117}$$

where the sum over repeated indices is understood. Here, $\frac{\delta(\chi\,|\,f)}{\delta g_{ij}(x)}$ is a distribution called the *functional derivative* of $(\chi\,|\,f)$ wrt $g_{ij}(x)$.

If two fields are canonically conjugate, *e.g.*, $q(x)$ and $p(x)$, then the Poisson brackets between two functionals $F[q,p]$, $G[q,p]$ on the phase space defined by q and p is

$$\{F,G\} = \int_\Sigma \mathrm{d}^3x \left(\frac{\delta F}{\delta q(x)} \frac{\delta G}{\delta p(x)} - \frac{\delta F}{\delta p(x)} \frac{\delta G}{\delta q(x)} \right) . \tag{118}$$

Notice that since one single field-theoretical constraint $\chi(x)$ is actually one constraint *per spatial point*, the Poisson bracket of χ with itself can be nonzero. One has in fact to smear it with two different smearing functions f and h, and the Poisson bracket will be

$$\{(\chi\,|\,f),(\chi\,|\,h)\} = \int_\Sigma \mathrm{d}^3x \left(\frac{\delta(\chi\,|\,f)}{\delta q(x)} \frac{\delta(\chi\,|\,h)}{\delta p(x)} - \frac{\delta(\chi\,|\,f)}{\delta p(x)} \frac{\delta(\chi\,|\,h)}{\delta q(x)} \right) , \tag{119}$$

which has no reason to vanish, unless χ is *ultralocal* (it contains no derivatives of q, p), in which case its variations will be linear in the smearings (with no derivatives acting on them), and by antisymmetry in f and h the expression above will vanish.

I will use the following notation for symmetrization and antisymmetrization of indices:

$$T^{[ij]} = \tfrac{1}{2}(T^{ij} - T^{ij})\,, \qquad T^{(ij)} = \tfrac{1}{2}(T^{ij} + T^{ij})\,.$$

The fundamental extended configuration space we will start with is Riem, the space of Riemannian 3-metrics. It is the geometrodynamical equivalent of Cartesian space $\mathsf{Q}^N = \mathbb{R}^{3N}$ for particle dynamics.

7 Relativity Without Relativity

In a series of papers Julian Barbour, Brendan Foster, and Niall Ó Murchadha and Edward Anderson and I [60, 61, 62, 63, 40], motivated by the desire to enforce reparametrization invariance and temporal relationalism, have shown how powerful the choice of a square-root form of the action is. This approach makes it possible, without any prior spacetime assumptions, to arrive at General Relativity, Special Relativity and gauge theory on the basis of relational first principles. I'll give here first a simplified account of the original results in the 'modern' language I used with E. Anderson in [40], and in Section 9 I'll get rid of all the simplifying assumptions made in this section and repeat the analysis in full generality (summarizing part of my recent contribution to this program with Edward Anderson [40]).

Let's start from the following assumptions:

1. *The action is a local functional of the 3-metric g_{ab} and its first derivatives.*
2. *The action is of Jacobi type, that is, the product of the square roots of a potential and a kinetic term which is quadratic in the velocities $\mathrm{d}g_{ab}$.*
3. *The theory must be free of any redundancy in the description of the fields, and this independence must be realized locally through best-matching.*

Then the simplest Lagrangian 'line element' satisfying these assumptions that one can write is[37]

$$\mathrm{d}\mathcal{L}_0 = \int \mathrm{d}^3x \sqrt{g}\sqrt{R - 2\,\Lambda}\sqrt{(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathrm{d}g_{ij}\,\mathrm{d}g_{kl}}\,, \tag{120}$$

where R is the 3-dimensional Ricci scalar and Λ is a spatially constant parameter. The relative factor between the two possible ways to contract the indices of the metric velocities $\mathrm{d}g_{ab}$ has been put to -1. This is the only working hypothesis and will be relaxed in the next subsection. It is important to notice that the above Lagrangian presupposes nothing about spacetime: it is merely the simplest parametrization-irrelevant and local Jacobi-type expression one can form from a 3-metric. No spacetime covariance (and no local Lorentz invariance) has been assumed. I'll show now how much can be deduced just from this. Calling the local expression

$$\mathrm{d}\chi = \frac{1}{2}\sqrt{\frac{(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathrm{d}g_{ij}\,\mathrm{d}g_{kl}}{R - 2\,\Lambda}}\,, \tag{121}$$

the field-theoretical version of the differential of the instant, the canonical momenta read

$$p^{ij} = \frac{\delta \mathrm{d}\mathcal{L}_0}{\delta \mathrm{d}g_{ij}} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathrm{d}g_{kl}\,. \tag{122}$$

[37] Notice that the square root is *inside* the integral. For this reason, the above action is not perfectly analogous to a Jacobi action for particle mechanics: in particle models the analogue of the integral $\int \mathrm{d}^3x$ is a sum over the particle index a and the vector component i: $S_{\text{Jacobi}} = 2\int [(E-U)\sum_{a,i} \frac{m_a}{2}\,(\mathrm{d}r^i_a)^2]^{1/2}$. This sum is inside the square root and makes the Jacobi action into a norm for the velocity vector $\mathrm{d}r^i_a$. The action $\int \mathrm{d}\mathcal{L}_0$ is not a proper norm, but putting the integral inside the square root by analogy with S_{Jacobi} would spoil the locality of the action. By locality I mean here that the action can be written as the integral of an expression that depends only on the values of the fields in a neighborhood of the point. As we shall see, this requirement of locality of the square root has very far-reaching consequences.

Due to the local square-root form of the action, the momenta satisfy at each space point the primary constraint

$$\mathcal{H} = \tfrac{1}{\sqrt{g}} \left(p^{ij} p_{ij} - \tfrac{1}{2} p^2 \right) - \sqrt{g} \, (R - 2\,\Lambda) \approx 0 \,, \tag{123}$$

where $p = p^{ij} g_{ij}$. The Euler–Lagrange equations are

$$\begin{aligned} \mathrm{d}p^{ij} = & \sqrt{g} \left(R \, g^{ij} - R^{ij} + \nabla^i \nabla^j - g^{ij} \Delta \right) \mathrm{d}\chi \\ & - 2\,\Lambda \, \sqrt{g} \, g^{ij} \, \mathrm{d}\chi - \frac{2\,\mathrm{d}\chi}{\sqrt{g}} \left(p^{ik} p_k{}^j - \tfrac{1}{2} p \, p^{ij} \right) \,. \end{aligned} \tag{124}$$

In order for the theory to be consistent, the constraint (123) must be propagated by the equations of motion,

$$\begin{aligned} \mathrm{d}\mathcal{H} & \approx 4 \, \mathrm{d}\chi \sqrt{g} \, \nabla_i \mathrm{d}\chi \, \nabla_j p^{ij} + 2 \, \mathrm{d}\chi \, \sqrt{g} \, \nabla_i \nabla_j p^{ij} \\ & = \frac{2}{\mathrm{d}\chi} \sqrt{g} \, \nabla_i \left(\mathrm{d}\chi^2 \, \nabla_j p^{ij} \right) \,. \end{aligned} \tag{125}$$

We see that the Hamiltonian constraint is not propagated unless $\nabla_j p^{ij} \approx 0$. This is not guaranteed and has to be imposed on the initial conditions through the additional constraint $\mathcal{H}_i = -2 \, \nabla_j p^j{}_i$. However, doing so doesn't ensure that the equations of motion will propagate the condition $\mathcal{H}_i \approx 0$: it has to be checked explicitly

$$\mathrm{d}\mathcal{H}_i = - \nabla_i \mathrm{d}\chi \, \mathcal{H} \approx 0 \,. \tag{126}$$

We see that this additional constraint is preserved by the evolution, as its propagation gives a linear combination of Hamiltonian constraints, which therefore vanishes weakly.

The new constraint we had to introduce is, as I already illustrated, the diffeomorphism constraint, and it acts on the metric as (307). Let's then realize diffeomorphism invariance at the level of the action through best matching:

$$\mathrm{d}\mathcal{L}_{\text{diff}} = \int \mathrm{d}^3 x \sqrt{g} \sqrt{R - 2\,\Lambda} \sqrt{ (g^{ik} g^{jl} - \, g^{ij} g^{kl}) \, \mathfrak{D} g_{ij} \, \mathfrak{D} g_{kl} } \,, \tag{127}$$

where the best-matching differential is

$$\mathfrak{D} g_{ij} = \mathrm{d} g_{ij} + \pounds_{\mathrm{d}\xi} g_{ij} = \mathrm{d} g_{ij} + \nabla_i \mathrm{d}\xi_j + \nabla_j \mathrm{d}\xi_i \,. \tag{128}$$

The local 'differential of the instant' [cf. (40)] and the canonical momenta have the same structure with the best-matching differential $\mathfrak{D}$ in place of d:

$$\mathrm{d}\chi = \frac{1}{2} \sqrt{ \frac{ (g^{ik} g^{jl} - g^{ij} g^{kl}) \, \mathfrak{D} g_{ij} \, \mathfrak{D} g_{kl} }{ R - 2\,\Lambda } } \,, \tag{129}$$

$$p^{ij} = \frac{\delta \mathrm{d}\mathcal{L}_{\text{diff}}}{\delta \mathrm{d} g_{ij}} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi} (g^{ik} g^{jl} - g^{ij} g^{kl}) \, \mathfrak{D} g_{kl} \,. \tag{130}$$

The local square-root form of the action still leads to (123) as a primary Hamiltonian constraint. The best-matching condition gives the diffeomorphism constraint

$$\frac{\delta \mathrm{d}\mathcal{L}_{\text{diff}}}{\delta \mathrm{d}\xi_i} = -2 \, \nabla_j \left[\frac{\sqrt{g}}{2\,\mathrm{d}\chi} (g^{ik} g^{jl} - g^{ij} g^{kl}) \, \mathfrak{D} g_{kl} \right] = -2 \, \nabla_j p^{ij} = \mathcal{H}^i \approx 0 \,. \tag{131}$$

The Euler–Lagrange equations, and in particular the term $\frac{\delta \mathrm{d}\mathcal{L}_{\text{diff}}}{\delta g_{ij}}$, are now changed, due to the dependence of the best-matching differential on the metric. This leads to the appearance of the term $\pounds_{\mathrm{d}\xi} p^{ij}$ term in

$$\begin{aligned} \mathrm{d}p^{ij} = & \sqrt{g} \left(\tfrac{1}{2} R \, g^{ij} - R^{ij} + \nabla^i \nabla^j - g^{ij} \Delta \right) \mathrm{d}\chi \\ & - 2\,\Lambda \, \sqrt{g} \, g^{ij} \, \mathrm{d}\chi - \frac{2\,\mathrm{d}\chi}{\sqrt{g}} \left(p^{ik} p_k{}^j - \tfrac{1}{2} p \, p^{ij} \right) \\ & + \pounds_{\mathrm{d}\xi} p^{ij} \,, \end{aligned} \tag{132}$$

but this term, as we have already seen in particle models, can be brought to the left to form a best-matching differential of p^{ij}:

$$\begin{aligned} \mathfrak{D} p^{ij} = \mathrm{d}p^{ij} - \pounds_{\mathrm{d}\xi} p^{ij} = & \sqrt{g} \left(\tfrac{1}{2} R \, g^{ij} - R^{ij} + \nabla^i \nabla^j - g^{ij} \Delta \right) \mathrm{d}\chi \\ & - 2\,\Lambda \, \sqrt{g} \, g^{ij} \, \mathrm{d}\chi - \frac{2\,\mathrm{d}\chi}{\sqrt{g}} \left(p^{ik} p_k{}^j - \tfrac{1}{2} p \, p^{ij} \right) \,. \end{aligned} \tag{133}$$

The same thing happens with the propagation of the Hamiltonian and diffeomorphism constraint:

$$\mathfrak{D}\mathcal{H} \approx \frac{2}{\mathrm{d}\chi} \sqrt{g} \, \nabla_a \left(\mathrm{d}\chi^2 \, \nabla_j p^{ij} \right) \,, \quad \mathfrak{D}\mathcal{H}_i = \mathrm{d}\chi \, \nabla_i \mathcal{H} - \nabla_i (\mathrm{d}\chi \mathcal{H}) \approx 0 \,, \tag{134}$$

The Hamiltonian constraint smeared with $\mathrm{d}\chi$ generates the same equations, forming Anderson's 'differential-almost-Hamiltonian':

$$\mathrm{d}\mathcal{A} = (\mathrm{d}\chi | \mathcal{H}) \,, \qquad \mathfrak{D} f = \{ \mathrm{d}\mathcal{A}, f \} \,, \tag{135}$$

together with the definition of the momenta (which is one of the two Hamilton equations),

$$\mathfrak{D} g_{ij} = \{ \mathrm{d}\mathcal{A}, g_{ij} \} = \frac{2\,\mathrm{d}\chi}{\sqrt{g}} \left(g_{ik} g_{jl} - \tfrac{1}{2} g_{ij} g_{kl} \right) \, p^{kl} \,. \tag{136}$$

Rigidity of the choice of the Lagrangian

In [63] the authors tested a somewhat more general ansatz for the potential term. They first considered an arbitrary power of the Ricci scalar, R^α, and then a linear combination of the terms R^2, $R^{ij}R_{ij}$ and ΔR, which are the only scalars with dimensions ℓ^{-4} that can be built with the metric field alone (R has dimensions ℓ^{-2} and no scalars with dimensions ℓ^{-3} exist). The propagation of the Hamiltonian constraint, in the words of the authors, "leads to an explosion of unpleasant non-cancelling terms", which rapidly end up trivializing the theory if included as new constraints (checking this explicitly is left as an exercise for the reader).

The Lagrangian (120) is not the most general possibility also in other respects: one could cancel the 'R' term, leaving the potential as a constant. Or one could change the relative factor between the two terms appearing in the kinetic term. These choices lead to interesting *and viable* alternatives. There was a preliminary discussion of them in [63] and in [60], but a thorough analysis of these cases has only recently been completed by Anderson and myself in [40]. I give a review of these results in Section 9. For the moment, I'll limit myself to noting an important detail, which at this level might seem unimportant and be ignored, but will become very relevant later. The issue regards the relative factor in the 'supermetric': if we generalize the Lagrangian (127) to

$$\mathrm{d}\mathcal{L}_{\text{diff-2}} = \int \mathrm{d}^3x\sqrt{g}\sqrt{R-2\,\Lambda}\sqrt{(g^{ik}g^{jl}-\lambda\, g^{ij}g^{kl})\,\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl}} \tag{137}$$

by adding the coefficient λ in the kinetic term, we get an additional term in the propagation of the Hamiltonian constraint:

$$\mathfrak{D}\mathcal{H} = \frac{2\sqrt{g}}{\mathrm{d}\chi}\nabla_i\left[\mathrm{d}\chi^2\,\nabla_j p^{ij} - \tfrac{\lambda-1}{3\lambda-1}\,\mathrm{d}\chi^2\nabla^i p\right]. \tag{138}$$

which, if the diffeomorphism constraint $\nabla_j p^{ij}\approx 0$ is already implemented, introduces a new constraint $p\approx const$. This is not propagated by the evolution, but it gives rise to a 'specifier' equation (case 3 of Dirac's analysis) which, in its turn, leads to a well-defined system with two propagating degrees of freedom (see Section 9 for the details). The new constraint has a simple geometric interpretation as the generator of position-dependent conformal transformations

$$g_{ab}\to\phi^4\,g_{ab}\,,\qquad \phi(x)>0\,, \tag{139}$$

(also called Weyl transformations) of the 3-metric. These transformations play an important role in York's solution of the initial-value problem of GR, which I review in Sec. 8, and, as I'll explain below, are needed to implement the Mach–Poincaré principle. It is striking that one ends up considering the same constraint $p\approx const$ in the solution of the initial-value problem and by considering a generalized supermetric.

Inclusion of a scalar field: Special Relativity

The ansatz for coupling of a scalar field to the metric field is

$$\begin{aligned}\mathrm{d}\mathcal{L}_{\text{scalar}} = \int \mathrm{d}^3x\sqrt{g}&\sqrt{R-2\,\Lambda+k\,g^{ij}\,\nabla_i\varphi\nabla_j\varphi+U(\varphi)}\\ &\cdot\sqrt{(g^{ik}g^{jl}-\,g^{ij}g^{kl})\,\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl}+\mathfrak{D}\varphi^2}\,,\end{aligned} \tag{140}$$

Here, $U(\varphi)$ can be any function of φ, and $\mathfrak{D}\varphi = \mathrm{d}\varphi+\pounds_{\mathrm{d}\xi}\varphi = \mathrm{d}\varphi+\mathrm{d}\xi^i\nabla_i\varphi$. A point to note here is that the form of the scalar kinetic term $\mathfrak{D}\varphi^2$ is, if assumed quadratic, uniquely fixed by best matching, while its coefficient is free because it can be changed by rescaling of the field. In contrast, the unknown constant k will then appear multiplying the field propagation term $g^{ij}\,\nabla_i\varphi\nabla_j\varphi$. This will have consequences, as we will soon see. Meanwhile, we see that the scalar field, together with the g_{ij} field, contributes directly to the local 'differential of the instant',

$$\mathrm{d}\chi = \frac{1}{2}\sqrt{\frac{(g^{ik}g^{jl}-g^{ij}g^{kl})\,\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl}+\mathfrak{D}\varphi^2}{R-2\,\Lambda+k\,g^{ij}\,\nabla_i\varphi\nabla_j\varphi+U(\varphi)}}\,, \tag{141}$$

but only indirectly (through $\mathrm{d}\chi$) to the metric momenta:

$$p^{ij} = \frac{\delta\mathrm{d}\mathcal{L}_{\text{scalar}}}{\delta\mathrm{d}g_{ij}} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}(g^{ik}g^{jl}-g^{ij}g^{kl})\,\mathfrak{D}g_{kl}\,. \tag{142}$$

The scalar field has its associated momentum

$$\pi = \frac{\delta\mathrm{d}\mathcal{L}_{\text{scalar}}}{\delta\mathrm{d}\varphi} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}\,\mathfrak{D}\varphi\,, \tag{143}$$

and the Hamiltonian constraint involves a quadratic combination of both the metric and the scalar-field momenta

$$\mathcal{H} = \sqrt{g}\,(R-2\,\Lambda+k\,g^{ij}\,\nabla_i\varphi\nabla_j\varphi+U(\varphi)) - \tfrac{1}{\sqrt{g}}\left(p^{ij}p_{ij}-\tfrac{1}{2}p^2+\pi^2\right)\approx 0\,. \tag{144}$$

The equations of motion are

$$\begin{aligned}\mathfrak{D}p^{ij} =& \sqrt{g}\left(\tfrac{1}{2}R\,g^{ij}-R^{ij}+\nabla^i\nabla^j-g^{ij}\Delta\right)\mathrm{d}\chi+k\,\nabla^i\varphi\nabla^j\varphi\,\sqrt{g}\,\mathrm{d}\chi\\ &+(k\,\nabla^k\varphi\nabla_k\varphi+U(\varphi)-2\,\Lambda)\sqrt{g}\,g^{ij}\,\mathrm{d}\chi-\frac{2\,\mathrm{d}\chi}{\sqrt{g}}\left(p^{ik}p_k{}^j-\tfrac{1}{2}p\,p^{ij}\right),\\ \mathfrak{D}\pi =& 2\,k\,\sqrt{g}\,\nabla_i\left(\nabla^i\varphi\,\mathrm{d}\chi\right)+\frac{\delta U}{\delta\varphi}\,\mathrm{d}\chi\,,\end{aligned} \tag{145}$$

and best-matching wrt diffeomorphisms gives

$$\frac{\delta\mathrm{d}\mathcal{L}_{\text{scalar}}}{\delta\mathrm{d}\xi_i} = -2\,\nabla_j p^{ij}+\pi\,\nabla^i\varphi = \mathcal{H}^i\approx 0\,. \tag{146}$$

Now I come to the key point: if we try to propagate the Hamiltonian constraint (144):

$$\begin{aligned}\mathfrak{D}\mathcal{H} &= \frac{2}{\mathrm{d}\chi}\sqrt{g}\,\nabla_i\left(\mathrm{d}\chi^2\,\nabla_j p^{ij}\right) + (4\,k+1)\frac{\sqrt{g}}{\mathrm{d}\chi}\nabla^i\left(\pi\,\nabla_i\varphi\,\mathrm{d}\chi^2\right)\\ &\approx (4\,k+1)\frac{\sqrt{g}}{\mathrm{d}\chi}\nabla^i\left(\pi\,\nabla_i\varphi\,\mathrm{d}\chi^2\right)\,,\end{aligned}\tag{147}$$

we find an obstruction. The option of introducing a new constraint $\pi\,\nabla_i\varphi \approx 0$ can be readily excluded, considering the fact that $\pi\,\nabla_i\varphi$ is a vector constraint which would kill 6 phase-space degrees of freedom whereas we introduced only two with the scalar field. The only remaining possibility is to propagate $\mathcal{H}$ strongly by setting $k = -\frac{1}{4}$. This result is very significant: we have found that the scalar field has to respect the same light cone as the metric field, which in turn implies Special Relativity in small regions of space for small intervals of time (local Lorentz invariance).

To see this, we treat φ as a test field that has no back reaction on the metric and consider a Euclidean patch, where the coordinates are chosen so that we can write the metric as a small perturbation around a static Euclidean metric $g_{ij} = \delta_{ij} + h_{ab}$. Since δ_{ij} is static, $\mathrm{d}\delta_{ij} = 0$ and $\mathrm{d}g_{ij} = \mathrm{d}h_{ij}$. We can put $\mathrm{d}\xi$ to zero as it is used to fix the coordinate gauge, and $\mathrm{d}\chi = \mathrm{d}t$ can be used as definition of the unit of time. The variation of Eq. (142) is

$$\begin{aligned}\frac{\mathrm{d}p^{ij}}{\mathrm{d}t} =& \tfrac{1}{2}\sqrt{g}(g^{ik}g^{jl} - g^{ij}g^{kl})\,\frac{\mathrm{d}^2 h_{kl}}{\mathrm{d}t^2}\\ &+\tfrac{1}{2}\sqrt{g}\left(g^{ij}\frac{\mathrm{d}h_{ij}}{\mathrm{d}t} + 2\frac{\mathrm{d}h^{ik}}{\mathrm{d}t}g^{jl} - 2\frac{\mathrm{d}h^{ij}}{\mathrm{d}t}g^{kl}\right)\frac{\mathrm{d}h_{kl}}{\mathrm{d}t}\,.\end{aligned}\tag{148}$$

The only first-order term is the first one, so

$$\frac{\mathrm{d}p^{ij}}{\mathrm{d}t} = \tfrac{1}{2}(\delta^{ik}\delta^{jl} - \delta^{ij}\delta^{kl})\,\frac{\mathrm{d}^2 h_{jl}}{\mathrm{d}t^2} + \mathcal{O}(h^2)\,.\tag{149}$$

The other side of the Euler–Lagrange equations is (since we are treating φ here as a test field with no backreaction on the metric, we ignore all the terms that depend on φ in the following equation)

$$\frac{\mathrm{d}p^{ij}}{\mathrm{d}t} = \left(\partial^k\partial^l h_{kl}\,\delta^{ij} - R^{ij}\right) - \tfrac{1}{2}(\delta^{ik}\delta^{jl} - \delta^{ij}\delta^{kl})\frac{\mathrm{d}h_{ij}}{\mathrm{d}t}\frac{\mathrm{d}h_{kl}}{\mathrm{d}t}\,,\tag{150}$$

where it's easy to show that

$$R_{ij} = \tfrac{1}{2}\left(\partial_j\partial^k h_{ik} + \partial_i\partial^k h_{jk} - \partial_k\partial^k h_{ij} - \delta^{kl}\partial_i\partial_j h_{kl}\right)\,.\tag{151}$$

Taken together, the above equations give the equations of motion of linearized gravity on a flat background. It is well known that these equations represent spin-2 gravitons propagating, in the coordinates we used, with a speed set to 1 [64].

Now consider the scalar field: the two terms of its Euler–Lagrange equations are

$$\frac{\mathrm{d}\pi}{\mathrm{d}t} = -2\,k\,\partial^i\partial_i\varphi + \frac{\delta U}{\delta\varphi} + \mathcal{O}(h^2)\,,\qquad \frac{\mathrm{d}\pi}{\mathrm{d}t} = \tfrac{1}{2}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\,,\tag{152}$$

which combined together give the following equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + 4\,k\,\partial^i\partial_i\varphi + 2\,\frac{\delta U}{\delta\varphi} = 0\,,\tag{153}$$

which is the equation of motion of a Klein–Gordon field with potential $2U$ *and propagation speed* $4\,k$. The choice of k that makes the theory consistent, $k = \frac{1}{4}$, is therefore the one that fixes the propagation speed of the scalar field to 1. *Thus, the scalar field respects the same light cone as the metric field.* Moreover, as is shown in [63] and we shall shortly show for a one-form, the mechanism that underlies this result is *universal*: it works for all fields coupled to the metric field.

This result is very striking. It is an independent derivation, from Machian first principles, *of the essence of Special Relativity*: constancy of the speed of light and a common light cone for all fields in nature. What is more, it implies local Lorentz invariance. Let me stress that General Relativity assumes the universal light cone and Lorentz invariance among its founding principles, while this theory *deduces them* from a smaller set of first principles, which are therefore arguably *more fundamental*.

Inclusion of a one-form field: Gauge Theory

The ansatz for inclusion of a one-form field is

$$\begin{aligned}\mathrm{d}\mathcal{L}_{\text{1-form}} = \int \mathrm{d}^3x\sqrt{g}&\sqrt{R - 2\,\Lambda + W(A) + V(g^{ij}A_iA_j)}\\ &\cdot\sqrt{(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl} + g^{ij}\mathfrak{D}A_i\mathfrak{D}A_j}\,,\end{aligned}\tag{154}$$

where $W(A) = \alpha\,\nabla_iA_j\nabla^iA^j + \beta\,\nabla_iA_j\nabla^jA^i + \gamma\,\nabla^iA_i\nabla^jA_j$, V can be any function of $g^{ij}A_iA_j$, and $\mathfrak{D}A_i = \mathrm{d}A_i + \pounds_{\mathrm{d}\xi}A_i = \mathrm{d}A_i + \mathrm{d}\xi^j\nabla_jA_i + \nabla_i\mathrm{d}\xi^jA_j$. The local differential of the instant is

$$\mathrm{d}\chi = \frac{1}{2}\sqrt{\frac{(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl} + g^{ij}\mathfrak{D}A_i\mathfrak{D}A_j}{R - 2\,\Lambda + W + V}}\,,\tag{155}$$

the metric momenta are

$$p^{ij} = \frac{\delta\mathrm{d}\mathcal{L}_{\text{1-form}}}{\delta\mathrm{d}g_{ij}} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathfrak{D}g_{kl}\,,\tag{156}$$

the momentum conjugate to A_a is

$$E^i = \frac{\delta \mathrm{d}\mathcal{L}_{\text{1-form}}}{\delta \mathrm{d}A_i} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}\, g^{ij}\, \mathfrak{D}A_j\,, \tag{157}$$

and the Hamiltonian constraint is

$$\mathcal{H} = \sqrt{g}\,(R - 2\,\Lambda + W + V) - \tfrac{1}{\sqrt{g}}\left(p^{ij}p_{ij} - \tfrac{1}{2}p^2 - g_{ij}E^i\,E^j\right) \approx 0\,. \tag{158}$$

The equations of motion are

$$\begin{aligned}
\mathfrak{D}p^{ij} =& \sqrt{g}\left(\tfrac{1}{2}R\,g^{ij} - R^{ij} + \nabla^i\nabla^j - g^{ij}\Delta + U + W\right)\mathrm{d}\chi \\
&+ 2\sqrt{g}\left(\alpha\,\nabla^k A^i \nabla_k A^j + \beta\,\nabla^k A^i \nabla^j A_k + \gamma \nabla^{(i}A^{j)}\,\nabla^k A_k\right)\mathrm{d}\chi \\
&- \sqrt{g}\,V'\,A^i\,A^j\,\mathrm{d}\chi - \frac{\mathrm{d}\chi}{\sqrt{g}}E^i\,E^j - \frac{2\,\mathrm{d}\chi}{\sqrt{g}}\left(p^{ik}p_k{}^j - \tfrac{1}{2}p\,p^{ij}\right)\,, \\
\mathfrak{D}E^i =& -2\sqrt{g}\,\left[\alpha\,\nabla_j(\nabla^j A^i\,\mathrm{d}\chi) + \beta\,\nabla_j(\nabla^i A^j\,\mathrm{d}\chi) + \gamma\,\nabla^i\,(\nabla^j A_b\,\mathrm{d}\chi)\right] \\
&+ 2\sqrt{g}\,V'\,A^i\,,
\end{aligned} \tag{159}$$

while best-matching wrt diffeomorphisms gives

$$\frac{\delta \mathrm{d}\mathcal{L}_{\text{1-form}}}{\delta \mathrm{d}\xi_i} = -2\,\nabla_j p^{ij} + E^j\,\nabla^i A_j - A^i\,\nabla_j E^j = \mathcal{H}^i \approx 0\,. \tag{160}$$

Propagation of the Hamiltonian constraint gives

$$\begin{aligned}
\mathfrak{D}\mathcal{H} =& \frac{\sqrt{g}}{\mathrm{d}\chi}\nabla_i\left(2\,\mathrm{d}\chi^2\,\nabla_j p^{ij} - \mathrm{d}\chi^2\,E^j\,\nabla^i A_j + \mathrm{d}\chi^2\,A^i\,\nabla_j E^j\right) \\
&- \frac{2\sqrt{g}}{\mathrm{d}\chi}\left[(\alpha + \tfrac{1}{4})\nabla^k(\mathrm{d}\chi^2\,E^j\,\nabla_k A_j) + (\beta - \tfrac{1}{4})\nabla^k(\mathrm{d}\chi^2\,E^j\,\nabla_j A_k)\right] \\
&- \gamma\frac{2\sqrt{g}}{\mathrm{d}\chi}\nabla^k(\mathrm{d}\chi^2\,E^k\,\nabla_j A^j) - \frac{\sqrt{g}}{\mathrm{d}\chi}\nabla^k\left(\mathrm{d}\chi^2\,\nabla_j E^j\,A_k\right)\,.
\end{aligned} \tag{161}$$

Here, the first line vanishes weakly due to the diffeo constraint (160). In the second and third lines there are three terms that offer us the option of making them vanish strongly through the choice of the parameters α, β and γ, otherwise they would imply the additional constraints

$$E^j\,\nabla_k A_j \approx const.\,, \quad E^j\,\nabla_j A_k \approx const.\,, \quad E^k\,\nabla_j A^j \approx const.\,. \tag{162}$$

None of these propagate, nor does any linear combination of them. Their propagation would thus require the introduction of further tertiary constraints, the procedure continuing (for any choice of α, β, γ) until we find an inconsistency. Therefore the only choice is to set $\alpha = -\frac{1}{4}$, $\beta = \frac{1}{4}$ and $\gamma = 0$. With this choice of parameters the W term in the potential takes the form

$$W = \tfrac{1}{4}(\nabla_i A_j \nabla^j A^i - \nabla_i A_j \nabla^i A^j)\,. \tag{163}$$

In (161) a further secondary constraint appears, this time without any option to kill it strongly:

$$\mathcal{G} = \nabla_i E^i \approx 0\,. \tag{164}$$

This is the 'Gauss constraint' of electromagnetism. Let's propagate it:

$$\mathfrak{D}\mathcal{G} = 2\sqrt{g}\,\nabla_i(\mathrm{d}\chi\,V'\,A^i)\,. \tag{165}$$

This shows that the only way to make it propagate is to take $V' = 0$, which means no potential – and, in particular, no mass term for the form field.

The secondary constraint we have found generates gauge transformations of A_i. Extending the definition of the Poisson brackets to include the form field and its conjugate momentum $\{F, G\} = \int \mathrm{d}^3x(\frac{\delta F}{\delta A_i}\frac{\delta F}{\delta E^i} - \frac{\delta F}{\delta E^i}\frac{\delta F}{\delta A_i})$, we get

$$\{A_i, (\sigma|\mathcal{G})\} = -\nabla_i\sigma\,. \tag{166}$$

We can easily implement this symmetry from the start, through best-matching.

Best-matching gauge transformations

The ansatz is

$$\begin{aligned}
\mathrm{d}\mathcal{L}_{\text{gauge}} = \int \mathrm{d}^3x\sqrt{g}&\sqrt{R - 2\,\Lambda + \tfrac{1}{4}(\nabla_i A_j \nabla^j A^i - \nabla_i A_j \nabla^i A^j)} \\
&\cdot\sqrt{(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl} + g^{ij}\mathfrak{D}A_i\mathfrak{D}A_j}\,,
\end{aligned} \tag{167}$$

with the modified best-matching differential

$$\mathfrak{D}A_i = \mathrm{d}A_i + \pounds_{\mathrm{d}\xi}A_i + \nabla_i \mathrm{d}\Phi\,. \tag{168}$$

With this choice everything is the same as above, except that now the momenta E^i have the modified definition

$$E^i = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}\,g^{ij}\,(\mathrm{d}A_i + \pounds_{\mathrm{d}\xi}A_i + \nabla_i \mathrm{d}\Phi)\,, \tag{169}$$

and we have to best-match with respect to $\mathrm{d}\Phi$,

$$\frac{\delta \mathrm{d}\mathcal{L}_{\text{gauge}}}{\delta \mathrm{d}\Phi} = -\nabla_i E^i\,. \tag{170}$$

We already know that the constraint algebra closes. The equations of motion for the form-field are

$$\mathfrak{D}E^i = \mathrm{d}E^i - \pounds_{\mathrm{d}\xi}E^i - \nabla_i \mathrm{d}\Phi = \tfrac{1}{2}\sqrt{g}\,\nabla_j(\nabla^j A^i \mathrm{d}\chi - \nabla^i A^j \mathrm{d}\chi)\,. \tag{171}$$

We see that the form field A_i enters the equations of motion only through the combination $F^{ij} = \nabla^i A^j - \nabla^j A^i$. We can use the dual vector field density

$$B^i = \tfrac{1}{2}\epsilon^{ijk}\nabla_j A_k\,, \tag{172}$$

(notice that the contravariant form of the Levi-Civita symbol ϵ^{ijk} is a density of weight 1, while the covariant form ϵ_{ijk} has weight -1). Then the equations of motion take the form

$$\mathrm{d}E^i - \pounds_{\mathrm{d}\xi} E^i - \nabla_i \mathrm{d}\Phi = \sqrt{g}\, \epsilon_{ijk} \nabla^j B^k \,, \tag{173}$$

but the vector density B^i satisfies a transversality constraint

$$\nabla_i B^i = 0 \,. \tag{174}$$

Let's now consider a flat background $g_{ij} = \delta_{ij}$ in Cartesian coordinates $\mathrm{d}\xi_i = 0$, and use the ephemeris time $\mathrm{d}\chi = \mathrm{dt}_{\mathrm{eph}}$. Equations (169), (170), (173) and (174) now take the form

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \,, \quad \frac{\mathrm{d}\mathbf{E}}{\mathrm{dt}_{\mathrm{eph}}} = -\boldsymbol{\nabla} \times \mathbf{B} \,, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \,, \quad \frac{\mathrm{d}\mathbf{B}}{\mathrm{dt}_{\mathrm{eph}}} = \boldsymbol{\nabla} \times \mathbf{E} \,, \tag{175}$$

These are obviously the source-free Maxwell equations. In addition we see that the definitions of B^i and of E^i in terms of A^i and its space and time derivatives are nothing more than the standard expressions of the magnetic and electric fields in terms of the vector potential:

$$\mathbf{E} = \tfrac{1}{2} \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{dt}_{\mathrm{eph}}} + \boldsymbol{\nabla} \mathrm{d}\Phi \right) \,, \qquad \mathbf{B} = \tfrac{1}{2} \boldsymbol{\nabla} \times \mathbf{A} \,. \tag{176}$$

Yang–Mills Theory

In [62] Anderson and Barbour considered the case of N 1-form fields coupled to each other in all possible ways compatible with a fairly general ansatz – namely, that the potential must be at most second order in the space derivatives and at most fourth order in the field variables. For the kinetic term, the only freedom that was left was to have a symmetric matrix coupling the best-matched velocities of the different 1-form fields. If A^α_i, $\alpha = 1, \dots, N$, are the various 1-form fields, propagation of the Hamiltonian constraint requires the introduction of N Gauss constraints:

$$\mathcal{G}_\alpha = \nabla_i E^i_\alpha - C^\beta{}_{\alpha\gamma}\, E^i_\beta\, A^\gamma_i \approx 0 \,, \tag{177}$$

and these non-Abelian gauge transformations must be implemented in the best-matching differential as

$$\mathfrak{D} A^\alpha_i = \mathrm{d}A^\alpha_i + \pounds_{\mathrm{d}\xi} A^\alpha_i + \nabla_i \mathrm{d}\Phi^\alpha + C^\alpha{}_{\beta\gamma}\, A^\beta_i\, \mathrm{d}\Phi^\gamma \,. \tag{178}$$

The kinetic term is therefore

$$\mathrm{d}\mathcal{L}^2_{\mathrm{kin}} = \delta_{\alpha\beta}\, g^{ij}\, \mathfrak{D} A^\alpha_i\, \mathfrak{D} A^\beta_j \,, \tag{179}$$

the potential term is constrained to be

$$W = g^{ik} g^{jl}\, \delta_{\alpha\rho} \left(\nabla_{[i} A^\alpha_{j]} + \tfrac{1}{2} C^\alpha{}_{\beta\gamma}\, A^\beta_i\, A^\gamma_j \right) \left(\nabla_{[k} A^\rho_{l]} + \tfrac{1}{2} C^\rho{}_{\sigma\tau}\, A^\sigma_k\, A^\tau_l \right) \,, \tag{180}$$

and the 3-index matrix $C^\alpha{}_{\beta\gamma}$ is constrained to satisfy the Jacobi rules

$$\delta_{\alpha\rho} \left(C^\alpha{}_{\beta\gamma}\, C^\rho{}_{\sigma\tau} + C^\alpha{}_{\beta\tau}\, C^\rho{}_{\sigma\gamma} + C^\alpha{}_{\beta\gamma}\, C^\rho{}_{\tau\sigma} \right) = 0 \,, \tag{181}$$

which implies, as Gell-Mann and Glashow have shown [65], that the matrix $C^\alpha{}_{\beta\gamma}$ represents the structure constants of a direct sum of compact simple and $U(1)$ Lie algebras. What we have found is nothing less than Yang–Mills theory, thus covering the whole bosonic sector of the Standard Model.

Further generalizations

In addition to the above, one could consider an antisymmetric tensor field $F^{ij} = -F^{ji}$ (the symmetric case is already included in the treatment of the metric field), but this turns out to be completely equivalent to treating the dual 1-form field $B_i = \frac{1}{2}\, \epsilon_{ijk}\, F^{jk}$, and similarly for the case of N antisymmetric tensor fields. The proof of these statements is left as an exercise.

The question posed in [62] of whether topological terms $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$ as proposed by 't-Hooft can be accommodated in this approach remains to be answered. Terms of this sort appear to be needed to explain the low-energy spectrum of QCD [66, 67], although they lead to the so-called 'strong CP problem' [68].

In [60, 61] Anderson extended the analysis to several new cases, for example Strong Gravity and generalizations thereof, in particular a new class of theories that can be considered as Strong-Gravity limits of Brans–Dicke theory. Particularly interesting (because they are realized in Nature) are spin-1/2 fermions of the theories of Dirac, Maxwell–Dirac and Yang–Mills–Dirac. Fermions are introduced in a somewhat phenomenological fashion, with the introduction of a spin connection but without a first-order formulation of the gravitational degrees of freedom. The kinetic fermionic part of the action is therefore outside the local square root, and it does not contribute to the differential of the instant. The potential term, instead, does contribute. A more satisfying treatment involves Palatini's formulation of GR, which does not have a local square root at all. A discussion of first-order formulations of gravity in the relational setting is still missing.

Finally, a paper [40] I recently wrote with E. Anderson completes the analysis of the metric field in full generality, moreover in a purely Hamiltonian setting, employing Dirac's method. These results will be summarized in Section 9, where they will be used to justify Shape Dynamics.

Further reading: The complete literature on the 'Relativity Without Relativity' approach is [63, 62, 60, 61, 40].

The problem of many-fingered time

The Machian first principles on which we have based our field theories up to now have proved to be quite powerful in not only deriving the relativity principle from a simpler set of axioms, but also in identifying all the kinds of (classical) fields that are presently thought to be fundamental in nature. However, these principles fail to realize, in the case of the gravitational field, what Barbour and Bertotti identified as the only precise formulation of Mach's principle: the Mach–Poincaré principle. The 'culprit' is the local square-root form of the action, the very thing that plays such a key role in some of the most interesting results I have shown: the derivation of the universal light cone and gauge theory. It does this because it leads to *local* Hamiltonian constraints, which constrain one degree of freedom at *each* space point. This leads to a mismatch. For having identified the diffeomorphism symmetry and implemented it through best-matching, we attributed to the gravitational field 3 degrees of freedom per space point, which is the dimensionality of Superspace, gravity's putative configuration space. But the dynamical laws we found do not realize the Mach–Poincaré principle on Superspace: given two points in it, there is a whole 'sheaf' of curves that extremalize our best-matching action (127). The variational principle we have, in its most advanced form (the 'differential-almost-Hamiltonian' approach), produces a curve in Superspace given initial data consisting of a point in Superspace and a transverse momentum, plus a lapse that depends on both time and space (check appendix A for the definition of lapse). But then any other lapse is equally good, and produces a different curve in Superspace which shares only the endpoints with the original one (see Fig. 16). The Einstein–Hilbert action assigns the same value to both curves. The uniqueness required by the Mach–Poincaré principle is absent. In Fig. 16 I show graphically what this ambiguity means: each curve in Superspace that can be generated by a different choice of lapse corresponds to a different foliation of the same spacetime.

From the Machian point of view, this situation is unacceptable, but the way out was actually anticipated already in the 1970's. As we can see from the counting of the degrees of freedom, the gravitational field has two Lagrangian degrees of freedom per point (Riem has six per point, the diffeos reduce it to Superspace, which has three, and the Hamiltonian constraint further reduces them to two). But Superspace has three degrees of freedom per space point, and therefore it cannot be the physical configuration space of GR. The Hamiltonian constraint, unlike the diffeo constraint, does not admit a simple geometrical interpretation in terms of transformations on the configuration-space variables alone, and therefore cannot be used to quotient Superspace into the physical configuration space of GR. The orbits of the Hamiltonian constraint do not lie on Riem: they mix the metric and the momenta. In other words, the vector field corresponding to the Hamiltonian constraint is not a map of Riem onto Riem: it lives inextricably in phase space, and the associated phase-space transformation of the metric depends on the momenta.

Moreover, the Barbour–Foster result [3] I sketched at the end of Sec. 6.2 strongly suggests that the Hamiltonian constraint is not *exclusively* a generator of gauge transformations. There ought to be a single linear combination of $\mathcal{H}(x)$, some $(f|\mathcal{H}) = \int \mathrm{d}^3x\, f(x)\, \mathcal{H}(x)$ which is associated with genuine evolution. If not, one would reach the absurd conclusion that dynamical evolution is a gauge transformation (look at Kuchař's review [69] for a criticism of such an idea). The particular linear combination of $\mathcal{H}$ that generates reparametrizations actually depends on the gauge-fixing we choose for $\mathcal{H}$, which in turn is related to the way we foliate spacetime. In a patch of spacetime, any positive function $f(x)$ can be used as a lapse defining a foliation, and the corresponding linear combination $(f|\mathcal{H})$ would then be our generator of true evolution.

The Hamiltonian constraint must be an admixture of gauge and dynamics, and, at the first glance, there does not appear to be any obvious way to disentangle the two if one insists that refoliation invariance is sacrosanct. As I said, a way out of this puzzle was already proposed in the 70's, mainly thanks to J. W. York, who built on the work of Y. Choquet-Bruhat and A. Lichnerowicz.

8 York's solution to the initial-value problem

I will now explain the core of York's work on the initial-value problem of GR. What York did is highly relevant for both Shape Dynamics and the Machian program in general: it indicated strongly that the physical degrees of freedom of GR are conformally invariant, in addition to diffeomorphism invariant, and it identified a corresponding fully reduced phase space of GR. As I will show, it allows us to find a solution to the problem of many-fingered time and to formulate a theory of geometrodynamics that satisfies the Mach–Poincaré principle and which we call Shape Dynamics.

York found a general method for solving the 'initial-value problem', which means finding phase-space data g_{ij}, p^{kl} on an initial Cauchy hypersurface that satisfy GR's Hamiltonian (305) and diffeomorphism constraints (306).

The diffeo constraint merely requires the momentum p^{ij} to be a tensor density whose covariant divergence wrt its conjugate g_{ij} vanishes. It is clearly closely analogous to the Gauss constraint of electrodynamics, which can also

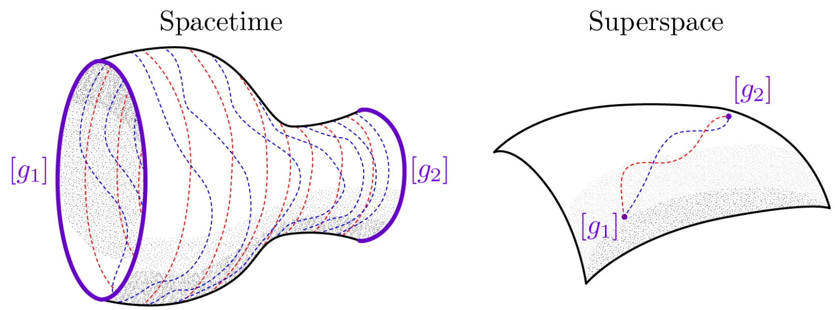

Figure 16: Many-fingered time issue: a single solution to Einstein's equations (the slab of spacetime on the left) between two Cauchy hypersurfaces (in purple) where two initial and final 3-geometries $[g_1]$, $[g_2]$ (the $[\,\cdot\,]$ brackets stand for 'equivalence class under diffeomorphisms') are specified, does not correspond to a single curve in superspace (on the right). Instead, for each choice of *foliation* of the spacetime on the left that is compatible with the boundary conditions $[g_1]$, $[g_2]$ there is a different curve in superspace. Here I have represented two different choices of foliation in red and a blue.

be formulated on a Riemannian manifold (Σ, g) and fixes the divergence of a vector field to be zero. In electrodynamics one can find an explicit solution of the Gauss constraint by identifying the longitudinal part of the vector field in terms of a scalar field that solves Poisson's equation. Subtraction of the longitudinal part then solves the problem. All of this is based on a very general result of differential geometry called the Helmholtz Decomposition Theorem, which allows one to identify the longitudinal and transverse components of a vector field on a general Riemannian manifold (see the first section of Appendix B.4).

Berger and Ebin [70] found an analogue of Helmholtz's theorem for symmetric 2-tensors (later rediscovered by Deser [71]), which allows one to uniquely determine their 'transverse' and 'longitudinal' parts (that is, the divergence-free part and the remainder) in terms of a vector field (analogous to the scalar field mentioned above) which must satisfy a certain elliptic equation, for which existence and uniqueness theorems hold. I won't reproduce this result here, since it is already contained as a sub-case in York's treatment of the diffeomorphism constraint, which I describe in detail in Appendix B.4. So the diffeomorphism constraint is no problem: it always admits a unique solution, at least on a compact manifold.[38]

The problem is the Hamiltonian constraint. With the method sketched above, I can build any solution I like of the diffeo constraint by starting with a 3-metric g_{ij} and an arbitrary symmetric tensor p^{ij} and extracting its transverse part wrt g_{ij} by solving an elliptic differential equation. Now this transverse part will not in general satisfy the Hamiltonian constraint: a quadratic combination of its components must be proportional to the Ricci scalar of g_{ij} at each point. The constraint is scalar, so I could try to locally rescale the tensor by a scalar function $p^{ij} \to f(x)\, p^{ij}$, adjusting its magnitude to solve the Hamiltonian constraint at each point, but then I would lose the transversality with respect to g_{ij}. Alternatively, I could consider the Hamiltonian constraint as a (nonlinear and very complicated) differential equation for g_{ij}, but the transverse part $p^{ij}_{\rm T}$ of p^{ij} is defined relative to g_{ij} itself: $p^{ij}_{\rm T}$ depends *nonlocally* on g_{ij}, and the equation becomes an integro-differential equation.

Lichnerowicz's partial solution

In 1944 [72] Lichnerowicz had found a way to decouple the Hamiltonian and diffeo constraint in the case of a *maximal* Cauchy hypersurface, that is, one whose extrinsic curvature has a vanishing trace $K = K^i{}_i = 0$. Consider the two constraints expressed in terms of the extrinsic curvature:

$$\nabla_b(K^{ij} - g^{ij}\, K) = 0\,, \qquad K^{ij}K_{ij} - K^2 - R = 0\,, \tag{182}$$

Lichnerowicz's strategy to find a metric g_{ij} and an extrinsic curvature K^{ij} that solve the above equations was to start from an arbitrary pair $\bar g_{ij}$, $\bar K^{ij}$ that do not solve it and make a conformal transformation of the metric

$$g_{ij} = \phi^4 \bar g_{ij}\,, \qquad g^{ij} = \phi^{-4}\bar g^{ij}\,, \qquad \phi \;\; \textit{smooth and positive}\,, \tag{183}$$

and simultaneously of the extrinsic curvature (see Appendix B.4) $K^{ij} = \phi^{-10}\bar K^{ij}$ that transform $\bar g_{ij}$, $\bar K^{ij}$ into g_{ij}, K^{ij} which do satisfy the constraints. Then the problem of satisfying the constraints become that of finding the correct ϕ. This is possible by virtue of the maximal slicing condition $K = 0$ and the ellipticity of the equation for ϕ that one finds. Let's see how this works.

As I show in Eq. (369) in Appendix B.4 below, under the conformal transformation

$$\nabla_b W^{ij} = \nabla_b(\phi^{-10}\bar W^{ij}) = \phi^{-10}\bar\nabla_j \bar W^{ij} - 2\,\phi^{-10}\,\bar W\,\partial^i \log\phi\,, \tag{184}$$

so, if $W^{ij} = K^{ij} - g^{ij}$, the diffeo constraint transforms to

$$\nabla_b(K^{ij} - g^{ij}\, K) = \phi^{-10}\bar\nabla_b \bar K^{ij} + \phi^{10}\partial^i(\phi^{-10}\,\bar K)\,, \tag{185}$$

[38] In the asymptotically flat case, one can find up to a 6-parameter family of solutions to the diffeo constraint associated with the Killing vectors of flat space (see Appendix B.4).This is done by using special boundary conditions and is important for the covariant definition of momentum and angular momentum of the gravitational field. However, it does not need to bother us at this stage.

and therefore if $K=0$ the transformed diffeo constraint does not depend on ϕ. It is this conformal covariance of the diffeo constraint in the $K=0$ case that makes it possible to decouple the two constraint equations and solve the Hamiltonian one separately just for the 'conformal factor' ϕ. The scalar curvature transforms according to the formula [72]

$$R = \phi^{-4}\,\bar{R} - 8\,\phi^{-5}\bar{\Delta}\phi\,. \tag{186}$$

Therefore the two constraints take the following form in the barred variables

$$\bar{\nabla}_b\,\bar{K}^{ij} = 0\,, \qquad \phi^{-12}\bar{K}^{ij}\bar{K}_{ij} - \phi^{-4}\,\bar{R} + 8\,\phi^{-5}\bar{\Delta}\phi = 0\,. \tag{187}$$

The last equation is strongly elliptic and quasilinear. For equations of this kind there are well-known theorems of existence and uniqueness of the solutions. The initial-value problem can be therefore solved with the following procedure:

1. Start with an arbitrary metric $\bar{g}_{ij}$ and an arbitrary traceless $\bar{K}^{ij}$.
2. Find the transverse part $\bar{K}^{ij}_{\mathrm{T}}$ of $\bar{K}^{ij}$ with respect to $\bar{g}_{ij}$.
3. Solve the 'Lichnerowicz equation' (187) for ϕ.
4. Then the 'physical' initial-value metric and extrinsic curvature, which satisfy the constraints, are $g_{ij} = \phi^4\bar{g}_{ij}$ and $K^{ij} = \phi^{-10}\bar{K}^{ij}_{\mathrm{T}}$.

Lichnerwicz's method has a serious limitation: on compact manifolds it works only for *Yamabe-positive* metrics (see below). The final physical data, in fact, satisfy the Hamiltonian constraint with $K=0$,

$$K^{ij}K_{ij} = R\,, \tag{188}$$

and the left-hand side is, by construction, non-negative. This implies that the techinque is only consistent if the scalar curvature on the right-hand side is non-negative *at every point* as well. This restricts the conformal equivalence class[39] of the metrics we can take as initial data.

To justify the last statement, I need to introduce a result in pure mathematics: the Yamabe theorem [73, 74], which Yamabe proposed as a conjecture and believed to have found its proof in 1960. However Trudinger found an error in Yamabe's proof in 1968, and later Aubin and Schoen supplied the correct proof around 1984. The theorem states that every Riemannian metric on a *closed manifold*[40] (of dimension ≥ 3) can be conformally transformed to a metric with *constant scalar curvature*. This constant is obviously not uniquely determined because one can change the magnitude of the Ricci scalar with a rescaling (a constant conformal transformation), but its sign is a conformal invariant. In fact, if the manifold is compact there cannot exist a conformal transformation mapping a metric g_{ab} with $R>0$ everywhere to another metric $\bar{g}_{ij}$ with $\bar{R}<0$ everywhere. To see this, take Eq. (186), multiply it by ϕ^{-1} and integrate it over all of Σ:

$$\int_\Sigma \sqrt{g}\,R\,\phi^{-1} = \int_\Sigma \sqrt{\bar{g}}\,(\bar{R}\,\phi - 8\,\bar{\Delta}\phi) \equiv \int_\Sigma \sqrt{\bar{g}}\,\bar{R}\,\phi\,. \tag{189}$$

The final step in which the Laplacian is cancelled is legitimate only if Σ is compact. The above equation proves that if R is definite everywhere it cannot change sign everywhere under a conformal transformation.[41] The sign of the Ricci scalar in the representation in which R is constant is therefore invariant under conformal transformations.

What is more, we can go further and introduce a real quantity, called the *Yamabe constant*, which is a conformal invariant and represents precisely the value of the Ricci scalar in the conformal gauge where it is a constant, rescaled by the volume of the manifold so that it's invariant under constant rescalings as well:

$$y[\Sigma, g] = \inf_\phi \left\{ \frac{\int \mathrm{d}^3x \sqrt{g}\,(\phi^2\,R - 8\,\phi\,\Delta\phi)}{\int \mathrm{d}^3x \sqrt{g}\,\phi^6} \right\}\,. \tag{190}$$

A theorem [77] by Yamabe, Trudinger, Aubin, and Schoen states that within a conformal equivalence class there exist metrics which realize the minimum, and they must have constant scalar curvature $R = \mathit{const.}$ Riemannian metrics can be classified according to the sign of the Yamabe constant, and divided into *Yamabe positive, negative* and *zero*, which means, according to the theorem, that they can be transformed into a metric with, respectively, positive, negative or zero constant scalar curvature. In general a manifold can be equipped with metrics belonging to different Yamabe classes. The following quantity:

$$\mathcal{Y}[\Sigma] = \inf_g y[\Sigma, g]\,, \tag{191}$$

is a topological invariant of the manifold Σ, and is called the *Yamabe invariant*. The Yamabe–Trudinger–Aubin–Schoen theorem states also that no manifold in dimension 3, 4 or 5 can have a Yamabe invariant larger than that of the 3-sphere:

$$\mathcal{Y}[\Sigma] \leq \mathcal{Y}[S^n]\,, \qquad n = 3, 4, 5\,. \tag{192}$$

[39] The conformal equivalence class, or conformal class for brevity, of a metric g_{ij} is defined as $\{\tilde{g}_{ij} / \exists\ \phi > 0, \phi \text{ smooth}, \tilde{g}_{ij} = \phi^4 g_{ij}\}$. The space of conformal classes of metrics is called *conformal superspace*, which in this tutorial we also refer to as shape space S when there is no risk of confusion with the shape space of the N-body problem. Shape Dynamics takes conformal superspace to be the reduced, physical configuration space of gravity.

[40] The analogous statement on noncompact manifolds is *wrong*, as proved by Jin [75].

[41] In the noncompact, asymptotically flat case one cannot discard the boundary value of $\Delta\phi$, and it is these boundary conditions that allowed York to define a mass-at-infinity for the gravitational field in the asymptotically flat case [76].

The manifolds themselves can then be classified according to the value of their Yamabe invariant: Type -1 manifolds only admit negative-curvature constant-curvature metrics. Yamabe type 0 admit constant-curvature metrics with zero or negative curvature. Type +1 manifolds admit at least one constant-curvature metric with positive curvature.

We now observe that the Lichnerowicz equation (187) is conformally covariant in the sense that if ϕ is the solution of (187) with data $\bar{g}_{ij}$, $\bar{K}^{ij}$, then for any smooth and positive θ one has that $\omega = \theta^{-1}\phi$ is the solution of (187) for the data $\tilde{g}_{ij} = \theta^4 \bar{g}_{ij}$, $\tilde{K}^{ij} = \theta^{-10}\bar{K}^{ij}$. Thus, one is free to choose the θ such that $\theta^4 \bar{g}_{ab}$ has Ricci scalar $R[\theta^4 \bar{g}_{ij}; x] = const$ with $const$ either $= -1, = 0$ or $= +1$ $\forall x$ (which exists by Yamabe's theorem). It is now evident that Lichnerowicz's method cannot be used with Yamabe-negative metrics on compact manifolds. For in that case one would have a conformal transformation (generated by ω) that maps a metric $\tilde{g}_{ij}$ with $R = -1$ to a metric g_{ab} which has an everywhere-positive R (because by assumption g_{ab} has to solve (188)). The Yamabe zero case is distinguished: in that case Lichnerowicz's equation can be solved only if $K^{ij} = 0$ at every point, as can be seen from (188). This is a very non-generic initial condition.

In the noncompact case, Eq. (188) admits a unique solution for any initial data if the manifold is asymptotically flat [78] (if the initial data include conditions at infinity).

An observation following the proof of the conformal covariance of Lichnerowicz's method: for practical reasons, one starts by specifying a complete 3-metric $\bar{g}_{ij}$ (six components) and a complete symmetric 2-tensor $\bar{K}^{ij}$ (another six components), but then the end result, in the form of g_{ab} and K^{ij}, depends only on the conformally-invariant and diffeo-invariant part of $\bar{g}_{ij}$, that is, its conformal class (two degrees of freedom). Moreover, it depends only on the TT-part of $\bar{K}^{ij}$, (which are another two degrees of freedom). These would be physical data that solve the initial-value problem if Lichnerowicz's solution were general, but it isn't. York succeeded in generalizing it to arbitrary Yamabe class, and can consequently claim to have identified the physical degrees of freedom of the gravitational field [79].[42] Let's see what he did.

York's general solution

In a series of papers [80, 79, 78, 81], York and Ó Murchadha made decisive progress by letting the extrinsic curvature have a spatially constant trace:

$$K^{ij} = K^{ij}_{\mathrm{TT}} + \tfrac{1}{3}\,\tau\, g^{ij}\,, \qquad \tau = const. \tag{193}$$

Here, τ is a spatial constant but it is time-dependent. In particular, it grows monotonically whenever York's method can be applied, and can be used as a time parameter in its own right. For this reason it is also referred to as the *York time*. A tensor such as (193) is automatically transverse with respect to the metric g_{ij} because the covariant divergence of a constant times g^{ij} is zero by the metric-compatibility condition. A further assumption of York makes all the difference. The transformation law of K^{ij} under conformal transformations is taken to be

$$K^{ij} = \phi^{-10}\,\bar{K}^{ij}_{\mathrm{TT}} + \tfrac{1}{3}\,\phi^{-4}\tau\,\bar{g}^{ij}\,, \tag{194}$$

so that after a conformal transformation the trace part is still spatially constant:

$$\bar{g}_{ij}\,\bar{K}^{ij} = g_{ij}\,K^{ij} \tag{195}$$

If that is the case, then Lichnerowicz's decoupling of the Hamiltonian and diffeo constraints continues to hold: the TT-part is conformally covariant, as proved in Appendix B.4, while the trace part is assumed to be invariant and therefore transverse as well:

$$\nabla_j(\tau\, g^{ij}) = 0\,, \qquad \bar{\nabla}_j(\tau\,\bar{g}^{ij}) = \nabla_j(\tau\,\bar{g}^{ij}) + \tau\,\bar{g}^{kj}\Delta\Gamma^i_{jk} + \tau\,\bar{g}^{ik}\Delta\Gamma^j_{jk} = 0\,, \tag{196}$$

where $\Delta\Gamma^a_{jk}$ is defined below. Thus when we pass from g_{ij}, K^{ij} to $\bar{g}_{ij}$, $\bar{K}^{ij}$ no term containing a derivative of ϕ appears in the diffeo constraint to threaten the transversality condition, which remains independent of ϕ, so that the two constraints are still decoupled. But now the conformally transformed Hamiltonian constraint gains a new term, quadratic in τ:

$$\phi^{-12}\,\bar{g}_{ik}\,\bar{g}_{jl}\,\bar{K}^{ij}_{\mathrm{TT}}\bar{K}^{kl}_{\mathrm{TT}} - \tfrac{2}{3}\tau^2 - \phi^{-4}\,\bar{R} + 8\,\phi^{-5}\bar{\Delta}\phi = 0\,. \tag{197}$$

Note the different powers of ϕ in front of the kinetic term and of $-\tau^2$, which is entirely due to York's assumption of the conformal invariance of τ. Equation (197) is called the Lichnerowicz–York equation. A solution of it exists and is unique on arbitrary compact or asymptotically flat manifolds, regardless of the conformal class of the metric. Let's see how one studies the solutions of Eq. (197). In accordance with the theory of quasilinear elliptic equations [82], Eq. (197) admits a unique solution *iff* the polynomial

$$f(z) = \tfrac{2}{3}\tau^2\,z^3 + R\,z^2 - KK\,, \tag{198}$$

(where I called $z = \phi^4$ and $KK = \bar{g}_{ik}\,\bar{g}_{jl}\,\bar{K}^{ij}_{\mathrm{TT}}\bar{K}^{kl}_{\mathrm{TT}}$) admits a single positive root at every point.

The function $f(z)$ has its extrema at $z = 0$ and $z = -R/\tau^2$, and its second derivatives at those points are $f''(0) = 2\,R$ and $f''(-R/\tau^2) = -2\,R$. Thus if $R > 0$, then $z = 0$ is a local minumum and $z = -R/\tau^2$ a local maximum; if $R > 0$, then vice versa. Moreover $f(0) = -KK$ is always negative, and changing the value of KK just shifts the whole function downwards. Therefore we always

[42] Subject to a caveat, which we shall mention below.

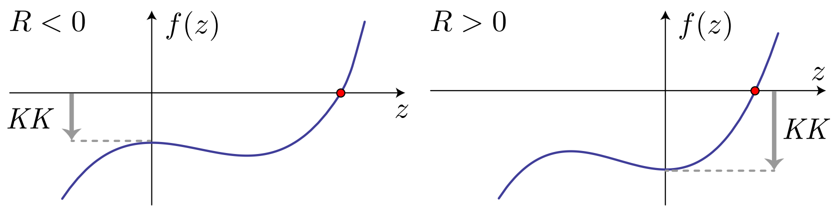

Figure 17: The polynomial (198) admits a single positive root for both signs of R. The term KK can only push the curve down, moving the root but not changing its nature.

fall into one of the two cases of Fig. 17, where we either have a maximum at $z=0$ and a minimum at $z>0$, with a zero to its right (for $R<0$), or we have a maximum at $z<0$ and a minimum at $z=0$, with a zero to its right (for $R>0$), all of this regardless of the value of KK. If we have $R=0$, the zero is at $z=(\frac{2}{3}KK/\tau^2)^{1/3}$, which is positive as long as $KK\neq 0$, otherwise it's zero.

York's general solution of the initial value problem works in a spacetime neighbourhood of a CMC (constant-mean-extrinsic curvature, meaning with $K = const.$) Cauchy hypersurface. Therefore the caveat promised in footnote 42 is that York's method can only be applied to CMC-foliable spacetimes, which however are a large class [83] and have nice singularity-avoidance properties.

Further reading: Lichnerowicz' paper [72], York's 1971 [80], 1972 [79] and 1973 [78] papers, Ó Murchadha's 'readings of the LY equation' [84], Misner–Thorne–Wheeler [85], Book on elliptic PDE's [82].

9 A first-principles derivation of SD

I will now bring to an end this Part's journey from relational first principles to Shape Dynamics. I will use the analysis which Edward Anderson and I made in [40], where we considered a pure-geometrodynamics theory in full generality, without any restriction on the potential term (apart from requiring a second-order potential which is covariant under diffeomorphisms), and completing the Dirac analysis of the constraints. Although SD was not discovered in [40], I find that analysis the most satisfactory way to justify it, starting from the present standpoint. For me, [40] represents the 'missing link' between the so-called 'Relativity Without Relativity' approach and modern Shape Dynamics. I will postpone to the beginning of the next section a review of the actual works that led to the formulation of SD.

The most general Jacobi-like, local square root Lagrangian we can take, with a lowest-order (dimension ℓ^{-2}) potential term is

$$\mathrm{d}\mathcal{L}_{\text{general}} = \int \mathrm{d}^3x\sqrt{g}\sqrt{a\,R-2\,\Lambda}\sqrt{(\lambda_1\,g^{ik}g^{jl}-\lambda_2\,g^{ij}g^{kl})\,\mathrm{d}g_{ij}\,\mathrm{d}g_{kl}}\,, \tag{199}$$

where a, Λ, λ_1 and λ_2 are spatially constant parameters. Since a global rescaling of the action by a constant is irrelevant, it is only the ratio between the parameters a and Λ that has a physical meaning. For this reason we take $a\in\{\pm1,0\}$, so that the parameter a only determines the sign of R and whether the corresponding term is present or not. The parameters λ_1, λ_2 present in the kinetic term parametrize the choice in the relative factor between the two possible ways to contract the indices of the metric velocities ($\lambda_1=\lambda_2=1$ is the 'DeWitt value', which corresponds to General Relativity).

The 'differential of the instant' is

$$\mathrm{d}\chi = \frac{1}{2}\sqrt{\frac{(\lambda_1 g^{ik}g^{jl}-\lambda_2\,g^{ij}g^{kl})\,\mathrm{d}g_{ij}\,\mathrm{d}g_{kl}}{a\,R-2\,\Lambda}}\,. \tag{200}$$

The canonical momenta are

$$p^{ij} = \frac{\delta \mathrm{d}\mathcal{L}_{\text{general}}}{\delta \mathrm{d}g_{ij}} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}(\lambda_1 g^{ik}g^{jl}-\lambda_2 g^{ij}g^{kl})\,\mathrm{d}g_{kl}\,. \tag{201}$$

Due to the local-square-root form of the action, they satisfy at each space point the Hamiltonian constraint

$$\mathcal{H} = \sqrt{g}\,(a\,R-2\,\Lambda) - \tfrac{1}{\lambda_1\sqrt{g}}\left(p^{ij}p_{ij} - \tfrac{\lambda_2}{3\lambda_2-\lambda_1}p^2\right) = 0\,. \tag{202}$$

Note that $\frac{\lambda_2}{3\lambda_2-\lambda_1}$ diverges for $\lambda_2=\frac{1}{3}\lambda_1$. This singular case requires special care because for it the momenta satisfy a further primary constraint.

The Euler–Lagrange equations are

$$\begin{aligned}\mathrm{d}p^{ij} =& a\sqrt{g}\left(\tfrac{1}{2}R\,g^{ij} - R^{ij} + \nabla^i\nabla^j - g^{ij}\Delta\right)\mathrm{d}\chi \\ &- 2\,\Lambda\sqrt{g}\,g^{ij}\,\mathrm{d}\chi - \frac{2\,\mathrm{d}\chi}{\lambda_1\sqrt{g}}\left(p^{ik}{p_k}^j - \tfrac{\lambda_2}{3\lambda_2-\lambda_1}p\,p^{ij}\right)\,.\end{aligned} \tag{203}$$

The same equations are generated by the total 'differential-almost-Hamiltonian' object

$$\mathrm{d}f = \{\mathrm{d}\mathcal{A},f\}\,, \qquad \mathrm{d}\mathcal{A} = (\mathrm{d}\chi|\mathcal{H})\,. \tag{204}$$

In order for the theory to admit solutions, the Hamiltonian constraint must be first-class with respect to itself, which, by virtue of (204), implies $\mathrm{d}\mathcal{H} \approx 0$, i.e., that it is propagated by the evolution. This is generated by the Poisson bracket

$$\{(\mathrm{d}\chi|\mathcal{H}),(\mathrm{d}\sigma|\mathcal{H})\} \approx \frac{a}{\lambda_1}\left(\mathrm{d}\chi\nabla^i\mathrm{d}\sigma - \mathrm{d}\sigma\nabla^i\mathrm{d}\chi\,\right| - 2\nabla_j p^j{}_i + 2\tfrac{\lambda_2-\lambda_1}{3\lambda_2-\lambda_1}\nabla_i p\Big). \quad (205)$$

We have a first set of choices here:

I. We can close the constraint algebra strongly by taking $a = 0$.

II. We can close the constraint algebra strongly by taking $\lambda_1 \to \infty$ (with λ_2 fixed).

III. We can introduce a new, secondary constraint

$$\mathcal{Z}_i = -2\,\nabla_j p^j{}_i + 2\,\alpha\nabla_i p\,, \quad \alpha = \tfrac{\lambda_2-\lambda_1}{3\lambda_2-\lambda_1}\,. \quad (206)$$

The first two possibilities correspond respectively to the removal of R (case **I**) and of the kinetic term (case **II**) from the Hamiltonian constraint.

Case I is known as *strong gravity*, a theory in which the light cones collapse to lines: no signal can be transmitted from one point to another, and each spatial point becomes causally disconnected from the others. This was named 'Carrollian relativity'[43] by Levy-Leblond [86, 87]. Belinsky, Khalatnikov and Lifshitz conjectured that near a cosmological singularity the contribution of matter to gravity becomes negligible compared with the self-coupling of gravity, and the variation of the gravitational field from one point to another can be neglected [88]. The strong gravity regime would then describe physics near a singularity according to this BKL conjecture.

Case II corresponds to a non-dynamical metric which is constant in time. The addition of matter gives rise to a dynamics in which signals are transmitted instantaneously across a fixed foliation. This is just Galilean relativity, which is the limit in which the speed of light goes to infinity.

Case III is more complicated, and divides into several sub-cases. First of all, on introducing a secondary constraint we have to check whether it closes a first-class system with itself and with the primary constraints.

The self-commutator of $\mathcal{Z}_i$ is

$$\{(\mathrm{d}\xi^i|\mathcal{Z}_i),(\mathrm{d}\chi^j|\mathcal{Z}_j)\} = ([\mathrm{d}\xi,\mathrm{d}\chi]^i|\ \mathcal{Z}_i + (2\,\alpha - 6\,\alpha^2)\nabla^i p)\,, \quad (207)$$

[43] After Lewis Carroll, whose Red Queen character in *Alice in Wonderland* says: "Now, here, you see, it takes all the running you can do, to keep in the same place".

where $[\mathrm{d}\xi,\mathrm{d}\chi]^a$ is the Lie bracket between the two vector fields $\mathrm{d}\xi$ and $\mathrm{d}\chi$. So the constraint closes on itself only if $6\,\alpha^2 - 2\alpha = 0$. This equation admits two solutions: $\alpha = 0$ (corresponding to the DeWitt value $\lambda_2 = -\lambda_1$) and $\alpha = \frac{1}{3}$. Both of the choices that make the constraint $\mathcal{Z}_i$ first-class wrt itself have a clear geometrical meaning. The choice $\alpha = 1$ makes $\mathcal{Z}_i \to \mathcal{H}_i = -2\,\nabla_j p^{ij}$ into the generator of diffeomorphisms, while $\alpha = \frac{1}{3}$ makes $\mathcal{Z}_i \to \mathcal{S}_i = -2\left(\nabla_j p^{ij} - \frac{1}{3}\nabla_i p\right)$ into the generator of *special diffeomorphisms*, or unit-determinant diffeomorphisms, which are diffeomorphisms that leave the local volume element $\sqrt{g}$ invariant.

Notice that special diffeomorphisms (strongly) commute with conformal transformations (generated by $p^{ij}g_{ij}$)

$$\{(\mathrm{d}\xi^i|\mathcal{S}_i),(\mathrm{d}\varphi|p)\} = 0\,, \quad (208)$$

where $\mathcal{S}_i$ corresponds to taking the traceless part of the momentum in the standard diffeo constraint,

$$(\mathrm{d}\xi^i|\mathcal{S}_i) = -2\int \mathrm{d}\xi_i \nabla_j (p^{ij} - \tfrac{1}{3}\,g^{ij}\,p)\,, \quad (209)$$

or the Lie derivative of the unimodular metric $g^{-\frac{1}{3}}g_{ij}$,

$$(\mathrm{d}\xi^i|\mathcal{S}_i) = \int p^{ij}\left(\pounds_{\mathrm{d}\xi} g_{ij} - \tfrac{2}{3}\,g_{ij}\nabla_k \mathrm{d}\xi^k\right) = \int p^{ij}\,g^{\frac{1}{3}}\,\pounds_{\mathrm{d}\xi}(g^{-\frac{1}{3}}g_{ij})\,. \quad (210)$$

Here $\pounds_{\mathrm{d}\xi}(g^{-\frac{1}{3}}g_{ij}) = \nabla_i \mathrm{d}\xi_j + \nabla_j \mathrm{d}\xi_i - \frac{2}{3}\,g_{ij}\nabla_k \mathrm{d}\xi^k$ is what York calls the *conformal Killing form* [78] associated with the vector $\mathrm{d}\xi$.

A related concept is that of *transverse* diffeos: they are diffeomorphisms generated by a transverse vector field $\nabla_i \xi^i = 0$. It's easy to show that $(\xi^i|\mathcal{H}_i) = (\xi^i|\mathcal{S}_i) - \frac{2}{3}(\nabla_i\xi^i|p)$; therefore if a vector field is transverse the diffeos it generates are special. The converse, however, is not true: there are special diffeomorphisms that are generated by vector fields with a longitudinal component.

Making a coordinate change from λ_1, λ_2 to α, λ_1 the Hamiltonian constraint takes the form

$$\mathcal{H} = \sqrt{g}\,(a\,R - 2\,\Lambda) - \tfrac{1}{\lambda_1\sqrt{g}}\left(p^{ij}p_{ij} - \tfrac{1}{2}\,(1-\alpha)\;p^2\right)\,, \quad (211)$$

Therefore the value $\alpha = 1/3$ corresponds to

$$\mathcal{H} = \sqrt{g}\,(a\,R - 2\,\Lambda) - \tfrac{1}{\lambda_1\sqrt{g}}\left(p^{ij}p_{ij} - \tfrac{1}{3}\,p^2\right)\,. \quad (212)$$

Analysis of case III

For any value of α other than 0 and $\frac{1}{3}$, we are forced to split $\mathcal{Z}_a$ into two secondary constraints: one is the diffeomorphism constraint $\mathcal{H}_i$, and the other should be a constraint implying $\nabla_i p = 0$.[44] A moment's thought reveals that the most generic possibility is to have $p = \frac{3}{2}\tau\sqrt{g}$, where τ is a spatial constant. The constraint p generates infinitesimal conformal transformations of the three-metric and its momenta:

$$\{(\varphi|p), g_{ij}\} = \varphi\, g_{ji}\,, \qquad \{(\varphi|p), p^{ij}\} = -\varphi\, p^{ij}\,. \tag{213}$$

The addition of the constant term τ changes the generated transformations into *volume-preserving conformal transformations* (VPCT's). In fact τ cannot be just any number: by consistency (if the base manifold Σ is compact, as I am assuming in this section) it is forced to be equal to (2 thirds of) the average of p over the whole space:

$$\tau = \tfrac{2}{3}\frac{\int \mathrm{d}^3x\, p}{\int \mathrm{d}^3x\sqrt{g}} \equiv \tfrac{2}{3}\langle p\rangle\,, \tag{214}$$

where the spatial average $\langle\,\cdot\,\rangle$ is defined as

$$\langle \rho\rangle := \frac{\int \mathrm{d}^3x\, \rho}{\int \mathrm{d}^3x\sqrt{g}} \tag{215}$$

for a scalar density ρ. Rewriting the constraint as

$$\mathcal{C} = p - \langle p\rangle\sqrt{g}\,, \tag{216}$$

we recognize its volume-preserving nature:

$$\begin{aligned}&\{g_{ab}, (\varphi|\mathcal{C})\} = (\varphi - \langle\varphi\rangle)\, g_{ij}\,,\\ &\{p^{ij}, (\varphi|\mathcal{C})\} = -(\varphi - \langle\varphi\rangle)(p^{ij} + \tfrac{1}{2}\langle p\rangle\sqrt{g}\, g^{ij})\,.\end{aligned} \tag{217}$$

The diffeo and conformal constraints close as a first-class system among themselves:

$$\begin{aligned}\{(\mathrm{d}\xi^i|\mathcal{H}_i), (\mathrm{d}\chi^j|\mathcal{H}_j)\} &= ([\mathrm{d}\xi, \mathrm{d}\chi]^i|\mathcal{H}_i)\,,\\ \{(\mathrm{d}\xi^i|\mathcal{H}_i), (\mathrm{d}\varphi|\mathcal{C})\} &= (\pounds_{\mathrm{d}\xi}\mathrm{d}\varphi|\mathcal{C})\,,\\ \{(\mathrm{d}\varphi|\mathcal{C}), (\mathrm{d}\rho|\mathcal{C})\} &= 0\,.\end{aligned} \tag{218}$$

Therefore case **III** divides into three sub-cases:

[44] Taking $\nabla_i p = 0$ itself as a new constraint would be wrong, as it is highly reducible (meaning that its components are not linearly independent) and we would be constraining too many (three) degrees of freedom. An equivalent scalar constraint is sufficient.

III.a $\boxed{\alpha = 0}$ Our secondary constraint is first-class wrt itself and generates regular diffeomorphisms. The diffeomorphism constraint $\mathcal{H}^i$ is propagated by $\mathcal{H}$:

$$\{(\mathrm{d}\chi|\mathcal{H}), (\mathrm{d}\xi^i|\mathcal{H}_i)\} = (\mathrm{d}\chi|\pounds_{\mathrm{d}\xi}\mathcal{H})\,, \tag{219}$$

and therefore we end up with a first-class system, which is just ordinary GR in the ADM formulation.

At this point we discovered our symmetries, so we can encode them back into the action through best-matching, as we did at the beginning of Sec. 7:

$$\begin{aligned}\mathrm{d}\mathcal{L}_{\text{diff}} &= \int \mathrm{d}^3x\sqrt{g}\sqrt{a\,R - 2\,\Lambda}\sqrt{(g^{ik}g^{jl} - g^{ij}g^{kl})\,\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl}}\,,\\ \mathfrak{D}g_{ij} &= \mathrm{d}g_{ij} + \pounds_{\mathrm{d}\xi}g_{ij} = \mathrm{d}g_{ij} + \nabla_i\mathrm{d}\xi_j + \nabla_j\mathrm{d}\xi_i\,,\end{aligned} \tag{220}$$

and the diffeo constraint becomes primary, a consequence of the free-endpoint variation wrt the best-matching field $\mathrm{d}\xi^i$.

III.b $\boxed{\alpha = 1/3}$ Our secondary constraint is first-class wrt itself and generates special diffeomorphisms. The propagation of the special diffeo constraint $\mathcal{S}_i$ gives

$$\{(\mathrm{d}\chi|\mathcal{H}), (\mathrm{d}\xi^i|\mathcal{S}_i)\} \approx -\tfrac{4}{3}a\,(\sqrt{g}(\Delta - R - 3\Lambda/a)\mathrm{d}\chi|\nabla^i\mathrm{d}\xi_i)\,. \tag{221}$$

This falls into case 3 of Dirac's analysis (see Sec. 6.2): it is a 'specifier' equation for the smearing $\mathrm{d}\chi$ of the Hamiltonian constraint. The system is second-class, and instead of each constraining two degrees of freedom per space point (giving a total of four physical dofs), the constraints $\mathcal{S}_i$ and $\mathcal{H}$ gauge-fix each other and constrain the dofs only down to six per point in total. The specifier equation is

$$\nabla_i\,[a(\Delta - R - 3\Lambda/a)\mathrm{d}\chi] = 0\,, \quad \Rightarrow \quad a\,(\Delta - R - 3\Lambda/a)\mathrm{d}\chi = \mathit{const.}\,. \tag{222}$$

The above equation is of the form $\Delta f(x) + g(x)\, f(x) + \mathit{const.} = 0$: it is an elliptic equation of the kind we have encountered in the preceding section, like the Lichnerowicz–York equation (197). It admits a unique solution for each positive root of the equation $g(x)\, f(x) + \mathit{const.} = 0$ (considered as an equation for $f(x)$ at each point), and there is only a one-parameter family of zeroes: in fact the constant represents an integration constant. One can write the equation as $(\Delta - R - 3\Lambda/a)\mathrm{d}\chi = \langle(\Delta - R - 3\Lambda/a)\mathrm{d}\chi\rangle$ or, since $\langle\Delta\mathrm{d}\chi\rangle = 0$,

$$\Delta\mathrm{d}\chi - (R + 3\Lambda/a)\mathrm{d}\chi + \langle(R + 3\Lambda/a)\mathrm{d}\chi\rangle = 0\,. \tag{223}$$

The last equation is homogeneous in $\mathrm{d}\chi$ and therefore the overall normalization of $\mathrm{d}\chi$ is irrelevant. This last one-parameter ambiguity is related to the fact that we can reparametrize the solution $\mathrm{d}\chi(x)$ with any time-dependent, spatially-constant function. It is the leftover reparametrization invariance of the solutions one obtains once $\mathcal{H}$ has been gauge-fixed. If $\mathrm{d}\chi_{\text{sol}}$ is the solution of the equation for $\mathrm{d}\chi$, then the generator of this reparametrization symmetry is

$$\mathrm{d}\mathcal{A}_{\text{global}} = (\mathrm{d}\chi_{\text{sol}}|\mathcal{H}) \approx 0\,, \tag{224}$$

which is the last leftover of the Hamiltonian constraint that hasn't been gauge-fixed by $\mathcal{S}_a$.

We ended up with a theory with six Hamiltonian degrees of freedom per point, those constrained by $\mathcal{S}_a \approx 0$, and a single, global Hamiltonian constraint $\mathrm{d}\mathcal{A}_{\text{global}}$, which can only be calculated by solving a differential equation (223). This equation is invariant under the transformations generated by $\mathcal{S}_a$,

$$\begin{aligned} g_{ij} &\to g_{ij} + \pounds_\zeta g_{ij} - \tfrac{2}{3}\, g_{ij} \nabla_k \zeta^k \\ p^{ij} &\to p^{ij} + \pounds_\zeta p^{ij} - \tfrac{2}{3} p^{ij} \nabla_k \zeta^k - \tfrac{1}{3}\, p\, \pounds_\zeta g^{ij}\,, \end{aligned} \tag{225}$$

(the proof of this is left as an exercise), and therefore the global Hamiltonian constraint is invariant as well.

Now, if we implement the special diffeomorphisms in the action through best-matching:

$$\begin{aligned} \mathrm{d}\mathcal{L}_{\text{Sdiff}} &= \int \mathrm{d}^3x \sqrt{g}\sqrt{a\,R - 2\,\Lambda}\sqrt{\left(g^{ik}g^{jl} - \tfrac{1}{3} g^{ij}g^{kl}\right)\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl}}\,, \\ \mathfrak{D}g_{ij} &= \mathrm{d}g_{ij} + \pounds_{\mathrm{d}\zeta} g_{ij} - \tfrac{2}{3}\, g_{ij}\nabla_k \mathrm{d}\zeta^k\,, \end{aligned} \tag{226}$$

the above action can be rewritten as

$$\begin{aligned} \mathrm{d}\mathcal{L}_{\text{Sdiff}} &= \int \mathrm{d}^3x \sqrt{g}\sqrt{a\,R - 2\,\Lambda}\sqrt{g^{ik}g^{jl}\mathfrak{D}g_{ij}\,\mathfrak{D}g_{kl}}\,, \\ \mathfrak{D}g_{ij} &= \mathrm{d}g_{ij} - \tfrac{1}{3} g_{ij} g^{kl} \mathrm{d}g_{kl} + \pounds_{\mathrm{d}\zeta} g_{ij} - \tfrac{2}{3}\, g_{ij}\nabla_k \mathrm{d}\zeta^k\,, \end{aligned} \tag{227}$$

and the relation between momenta and velocities, $p^{ij} = \frac{\sqrt{g}}{2\,\mathrm{d}\chi}(g^{ik}g^{jl} - \frac{1}{3} g^{ij}g^{kl})(\mathrm{d}g_{ij} + \pounds_{\mathrm{d}\zeta} g_{ij})$, is not invertible. Therefore the momenta satisfy an additional primary constraint: $g_{ij}p^{ij} = 0$. We are forced anyway to include conformal transformations: this case therefore reduces to a sub-case of **III.c**, which I now discuss.

III.c $\boxed{\text{generic } \alpha}$ We have to introduce *two* separate secondary constraints, the diffeomorphism constraint $\mathcal{H}_a$ and the volume-preserving conformal constraint $\mathcal{C} = p - \langle p \rangle \sqrt{g}$. These constraints are first-class among themselves. As in the last case, the propagation of $\mathcal{C}$,

$$\begin{aligned} \{(\mathrm{d}\chi|\mathcal{H}), (\mathrm{d}\sigma|\mathcal{C})\} \approx 2\,a(&\sqrt{g}(\Delta - R - 3\Lambda/a)\mathrm{d}\chi \\ &- \tfrac{1-3\,\alpha}{2\lambda_1}\langle p \rangle\, p\, \mathrm{d}\chi - \frac{3}{2}\,\mathrm{d}\chi\mathcal{H}\,|\,\mathrm{d}\sigma - \langle \mathrm{d}\sigma \rangle)\,. \end{aligned} \tag{228}$$

gives a 'specifier' equation for $\mathrm{d}\chi$,

$$2\,a\,(\Delta - R - 3\Lambda/a)\mathrm{d}\chi - \tfrac{1-3\,\alpha}{2\lambda_1}\,\langle p \rangle^2\,\mathrm{d}\chi = const.\,. \tag{229}$$

This again is an elliptic equation and admits a unique solution. This solution $\mathrm{d}\chi_{\text{sol}}$, if used to smear the Hamiltonian constraint, $(\mathrm{d}\chi_{\text{sol}}|\mathcal{H})$, gives the part of $\mathcal{H}$ that is first-class with respect to the corresponding conformal constraint. This first-class part is a single residual global Hamiltonian constraint

$$\mathrm{d}\mathcal{A}_{\text{global}} = (\mathrm{d}\chi_{\text{sol}}|\mathcal{H}) \approx 0\,. \tag{230}$$

This $\mathrm{d}\mathcal{A}_{\text{global}}$ is invariant under VPCT's (and diffeomorphisms), and therefore generates the dynamics in the reduced configuration space.

The theory we obtained satisfies the strong Mach–Poincaré principle in the quotient of Superspace by VPCT's. The quotient of Superspace by ordinary conformal transformations is *conformal superspace*, which we also call *Shape Space* S in analogy to the particle models:

$$\mathsf{S} := \mathsf{Superspace}/\mathsf{Sim}\,, \qquad g_{ij} \sim g'_{ij} \quad \text{iff} \;\; \exists \;\; \phi \quad s.t. \;\; g_{ij} = e^{4\phi} g'_{ij}\,. \tag{231}$$

The quotient of Superspace by VPCT is just[45] the Cartesian product between S and the real positive line $\mathbb{R}^+$, representing the dof associated with the total volume of space, $V = \int \mathrm{d}^3x\sqrt{g}$.

These results do not depend on the value of α, and therefore they hold also if $\alpha = 0$ or 1/3. The sub-cases **III.a** and **III.b** are included in this one. The theory we are dealing with is (generalized) Shape Dynamics. I call it generalized because proper SD refers just to the case $\alpha = 0$. It is not clear, at this point, whether the value of α has physical meaning. This is because in the shape-dynamical description its value can be reabsorbed into a rescaling of York time. Should it turn out that α has physical meaning, then it would have to be considered as a dimensionless coupling for Shape Dynamics. The chances then are that this coupling would run under Renormalization Group Flow, and $\alpha = 0$ would presumably be the IR value, where equivalence with General Relativity (and with it spacetime covariance) emerges as a sort of accidental IR symmetry. The $\alpha = 1/3$ value might then be the UV limit.

[45] Locally - one has to exclude degenerate metrics and have special care in the case of metrics with conformal isometries

This table summarizes the different cases we have encountered:

Case	Parameter values	DOF's	Theory
I	$a = 0$	4-12 p.p.	Carroll Relativity
II	$y = 0$	0	Galilean Relativity
III.a	$\alpha = 0$	4 p.p.	General Relativity
III.b	$\alpha = \frac{1}{3}$	4 p.p.+ 2 global	???
III.c	any	4 p.p. + 2 global	Shape Dynamics

I have reached the point at which I can finally introduce Shape Dynamics. As Henneaux and Teitelboim state in their book [51], a second-class system like that of case **III.c** can be seen as a gauge-fixing of a first-class system. Doing this often requires enlargement of the phase space with further redundant (constrained) degrees of freedom, and this is the case also in Shape Dynamics, where it is necessary to introduce a scalar field ϕ and its conjugate momentum π. Then one modifies the Hamiltonian, diffeo and VPCT constraints:

$$\begin{aligned}\mathcal{H} =& \frac{e^{-6\hat{\phi}}}{\sqrt{g}} \left(p^{ab} - \tfrac{1}{3} p g^{ij}\right)\left(p_{ij} - \tfrac{1}{3} p g_{ij}\right) - (1-\alpha)\frac{e^{-6\hat{\phi}}}{6\sqrt{g}} \left(p - \sqrt{g}\left(1 - e^{-6\hat{\phi}}\right)\langle p\rangle\right)^2 \\ & - \sqrt{g}\, R\, e^{2\,\phi} - \sqrt{g}\, e^{\phi} \Delta e^{\phi} \approx 0\,, \\ \mathcal{H}^i =& -2\,\nabla_j p^{ij} + \pi\,\nabla^i \phi \approx 0\,, \qquad \mathcal{Q} = \pi - 4\left(p - \langle p\rangle \sqrt{g}\right) \approx 0\,,\end{aligned}$$

The above system is first-class. This theory is called the *Linking Theory*, and it leads to SD as the following gauge-fixing: $\pi \approx 0$. That generates the set of constraints we found in case **III.a** (for $\alpha = 0$). A different gauge-fixing, namely $\phi \approx 0$, gives instead a first-class system (General Relativity) by killing just the modified VPCT constraint $\mathcal{Q}$. I'll define from scratch, and more carefully, the Linking Theory in Sec. 11 after some historical background.

Further reading: my paper with E. Anderson [40].

Part IV

Shape Dynamics

10 Historical Interlude

In Sec. 11, I will present Shape Dynamics as the first-class extension in the manner of Henneaux and Teitelboim [51] of the theory we studied in case **III.c** of last Section, when $\alpha = 0$. However, that's not how SD was originally discovered, which I now briefly recount before proceeding.

The stimulus came from Barbour's desire to create a scale-invariant Machian theory, first of particle dynamics and then dynamical geometry. The first step was the derivation in 1999 of a particle-dynamics model (published in 2003 [43]) with dilatational best matching and the Newton gravitational potential V_{New} (of degree -1) replaced by the potential $I_{\text{cm}}^{-1/2} V_{\text{New}}$, where I_{cm} is the centre-of-mass moment of inertia. The dilatational constraint $D = \sum_a m_a \mathbf{r}_a \cdot \mathbf{r}_a$ resulting from the dilatational best matching commutes with this potential, and the resulting theory defines geodesics on shape space. The strong Mach–Poincaré principle is therefore satisfied.

Barbour and Ó Murchadha then extended the underlying ideas of this particle model to dynamical geometry in [89] in 1999. This proposed and implemented the idea that the refoliation invariance of GR should be replaced by invariance under full three-dimensional conformal transformations. This led to a geodesic theory on conformal superspace that satisfies the strong Mach–Poincaré principle and eliminates a perceived Machian defect of GR highlighted by York's work on its initial-value problem. He had shown that it could be solved by taking the following initial data: a conformal equivalence class of 3-geometries, its variation, and a single real quantity (the value of York time or, as Ó Murchadha showed [84], the spatial volume). The necessity for this additional degree of freedom is quite puzzling. One could completely specify a solution of GR with initial data on conformal superspace S (two degrees of freedom per point) if it were not for this additional single global degree of freedom, which is not conformally invariant. This puzzle is eliminated in the theory of [89], which was later more fully developed in [43] and is based on a modified form of best matching in which the action is not equivariant. With ϕ the best-matching field, it depends not only on $\mathrm{d}\phi$, as in the original best-matching proposed by Barbour and Bertotti and but also, in a decisive innnovation due to Ó Murchadha, on ϕ. One has to vary independently wrt ϕ and $\mathrm{d}\phi$.

The work on this theory was interrupted by the successes of the Relativity Without Relativity approach but then explored further by Barbour, Foster, Ó Murchadha, Anderson and Kelleher. However, it soon became clear that although the theory had a good chance to reproduce the predictions of GR for local subsystems of the Universe it would badly fail to explain cosmological observations because in it the volume of the Universe must remain constant. Interest was then concentrated on a theory invariant under volume-preserving conformal transformations (VPCTs). Such transformations change the ratios of the determinant g of the metric freely at different space points but leave the total volume of space unchanged. It had already been conjectured in [89] that such a theory would be observationally equivalent to GR, but progress was slow until Foster found a neat and computationally covenient way to implement VPCTs. The transformation takes the form

$$g_{ij} \to e^{4\hat{\phi}} g_{ij}\,, \qquad \hat{\phi}(x) := \phi(x) - \tfrac{1}{6} \log \langle \sqrt{g}\, e^{6\phi} \rangle\,, \tag{232}$$

in which ϕ is subject to no restriction except $\phi > 0$ and would implement an unrestricted conformal transformation were it not for the correction $-\frac{1}{6}\log\langle\sqrt{g}\,e^{6\phi}\rangle$, which restores the total volume to the value it had before the transformation implemented by the unrestricted first term. This device then rapidly led to the papers [90, 91], which created a VPCT-invariant theory based on the modified form of best matching employed in [89, 43].

Because of the use of VPCTs, this meant that the reduced configuration space of the theory is Conformal Superspace + Volume (CS+V), so that one still has the curious extra (non-shape) degree of freedom, but the use of VPCTs, as opposed to unrestricted conformal transformations led to a crucial result: it patched a worrisome flaw in York's method: When attributing a constant trace (times $\sqrt{g}$) to the momentum:

$$g_{ij} p^{ij} = \sqrt{g}\, \tau\,, \tag{233}$$

York did it in a particular conformal frame, which was not the one in which the metric and the momenta satisfy the Hamiltonian constraint. But when making connection to that frame, York assumed that the trace of the momentum transforms like $\sqrt{g}$ under confromal transformations, that is, he assumed the York time τ to be invariant! York did not give a satisfactory justification of this assumption in his work. But [90] found a first-principles justification of what the authors called 'York scaling': first, York's 'CMC' condition should be understood as the nonlocal expression:

$$g_{ij} p^{ij}(x) = \sqrt{g(x)}\, \langle p \rangle = \sqrt{g(x)} \frac{\int d^3y\, p(y)}{\int d^3z \sqrt{g(z)}}\,; \tag{234}$$

second, the conformal transformations to the frame in which the scalar constraint is satisfied should only be *volume-preserving* and implemented as in (232). They then have a nontrivial action on the canonical momenta:

$$p^{ij} \to e^{-4\hat{\phi}} \left[p^{ij} - \tfrac{1}{3}(1 - e^{6\hat{\phi}}) \langle g_{kl} p^{kl} \rangle \sqrt{g}\, g^{ij} \right]\,, \tag{235}$$

and one can verify that this transformation leaves the York time, written as $\langle p \rangle$, invariant (left as an exercise). This was an important conceptual clarification brought about by [90, 91]. However, the variational principle behind the 'non-equivariant best matching' used in these papers was not adequately explained and, moreover, did not lead to a genuine gauge theory of conformal geometrodynamics but rather to a representation of GR in CMC gauge. It was several years before the next step was taken. To this we now turn.

11 Shape Dynamics and the Linking Theory

In [92, 93], Gomes, Gryb and Koslowski formulated Shape Dynamics the way we understand it now, and they sharpened the idea of intersecting constraint surfaces with a global flow in the intersection which I illustrated in Fig. 4. In the formulation of [92, 93] Shape Dynamics is founded on GR, and on the realization that in the same phase space of ADM two first-class systems of constraints coexist which are dual to each other in the sense that they represent good gauge-fixings of each other. The proper treatment of this system in the Hamiltonian language makes the mathematics of [92, 93] more solid than that of [90, 91], however, from a foundational point of view, these papers lack the self-contained logic which characterized the RWR approach: they relied on GR as the construction principle for SD. It was for that reason that I introduced SD with last section's analysis, which first appeared in [40], in order to provide the 'missing link' between the RWR approach and modern Shape Dynamics.

So let's consider now the 'Linking Theory', which is the first-class extension of the second-class system of case **III.c**, when $\alpha = 0$.

The most elegant way to introduce the Linking Theory is to keep explicit equivalence with GR at every stage. So we start with the ADM system:

$$\begin{aligned} \mathcal{H} &= \frac{1}{\sqrt{g}} \left(p^{ij} p_{ij} - \tfrac{1}{2} p^2 \right) - \sqrt{g}\, R\,, \\ \mathcal{H}^i &= -2\, \nabla_j\, p^{ij} \approx 0\,. \end{aligned} \tag{236}$$

Then we trivially extend the phase space (the cotangent bundle to $\mathsf{Riem}(\Sigma)$) with a scalar field ϕ and its conjugate momentum π. We also add a further constraint:

$$\mathcal{Q} = \pi \approx 0\,, \tag{237}$$

which makes ϕ into a gauge degree of freedom. This constraint is trivially first-class with respect to the other ones (236). So we have a first-class system which has the same number of degrees of freedom as ADM gravity and is trivially equivalent to it: ADM gravity can be recovered with the gauge fixing $\phi \approx 0$.

Now we perform a canonical transformation with a type-2 generating functional:

$$\begin{aligned} \mathrm{F} &= \int \mathrm{d}^3x \left(g_{ij}\, P^{ij} + \phi\, \Pi + g_{ij} (e^{4\,\hat\phi} - 1) P^{ij} \right) , \\ \hat\phi(x) &:= \phi(x) - \tfrac{1}{6} \log \langle \sqrt{g}\, e^{6\phi} \rangle \,. \end{aligned} \tag{238}$$

The transformation rules are

$$\begin{aligned} \frac{\delta \mathrm{F}}{\delta g_{ij}} &= e^{4\,\hat\phi}\, P^{ij} + \tfrac{1}{3}(e^{6\hat\phi} - 1) \langle e^{4\,\hat\phi}\, g_{kl} P^{kl} \rangle_g \sqrt{g}\, g^{ij} \,, & \frac{\delta \mathrm{F}}{\delta \Pi^{ij}} &= e^{4\,\hat\phi}\, g_{ij} \,, \\ \frac{\delta \mathrm{F}}{\delta \phi} &= \Pi + 4 \left(e^{4\,\hat\phi}\, g_{ij}\, P^{ij} - \langle e^{4\,\hat\phi}\, g_{ij}\, P^{ij} \rangle_g \sqrt{g}\, e^{6\,\hat\phi} \right) , & \frac{\delta \mathrm{F}}{\delta \Pi} &= \phi \,, \end{aligned} \tag{239}$$

which translate into $(g_{ij}, \phi; p^{ij}, \pi) \to (G_{ij}, \Phi; P^{ij}, \Pi)$, where

$$\begin{aligned} P^{ij} &= e^{-4\,\hat\phi} \left[p^{ij} - \tfrac{1}{3}(1 - e^{6\hat\phi}) \langle p \rangle \sqrt{g}\, g^{ij} \right] , & G_{ij} &= e^{4\,\hat\phi}\, g_{ij} \,, \\ \Pi &= \pi - 4\,(p - \langle p \rangle \sqrt{g}) \,, & \Phi &= \phi \,. \end{aligned} \tag{240}$$

Now express the Hamiltonian, diffeo and $\mathcal{Q}$ constraints in terms of the transformed variables [this calculation is left as an exercise – don't forget that the covariant derivative of a tensor density of weight $w = 1$ is $\nabla_k p^{ij} = \partial_k p^{ij} + \Gamma^i_{kl} p^{lj} + \Gamma^j_{kl} p^{il} - \Gamma^l_{kl} p^{ij}$ and that the Christoffel symbols transform under conformal transformations $g_{ij} \to e^{4\phi}$ as $\Gamma^i_{jk} \to \Gamma^i_{jk} + 2(\delta^i{}_j \partial_k \phi + \delta^i{}_k \partial_j \phi - g_{jk} g^{il} \partial_l \phi)$], and obtain

$$\begin{aligned} \mathcal{H}_{\hat\phi} =& \frac{e^{-6\hat\phi}}{\sqrt{g}} \left(p^{ij} p_{ij} + \frac{1}{3} \sqrt{g} \left(1 - e^{6\hat\phi} \right) \langle p \rangle\, p - \frac{1}{6} g \left(1 - e^{6\hat\phi} \right)^2 \langle p \rangle^2 - \frac{p^2}{2} \right) \\ & - \sqrt{g} \left(R\, e^{2\,\hat\phi} - 8\, e^{\hat\phi} \Delta e^{\hat\phi} \right) \approx 0 \,, \\ \mathcal{H}^i_{\hat\phi} =& -2\, e^{-4\hat\phi} \left[\nabla_j p^{ij} - 2\,(p - \sqrt{g}\,\langle p \rangle)\, \nabla^i \phi \right] \approx 0 \,, \\ \mathcal{Q}_{\hat\phi} =& \pi - 4\,(p - \langle p \rangle \sqrt{g}) \approx 0 \,, \end{aligned} \tag{241}$$

which are equivalent to (another exercise for the reader)

$$\begin{aligned} \mathcal{H}_{\hat\phi} =& \frac{e^{-6\hat\phi}}{\sqrt{g}} \left(p^{ij} - \tfrac{1}{3} p g^{ij} \right) \left(p_{ij} - \tfrac{1}{3} p g_{ij} \right) - \frac{e^{-6\hat\phi}}{6\sqrt{g}} \left(\frac{\pi}{4} + e^{6\hat\phi} \sqrt{g}\, \langle p \rangle \right)^2 \\ & - \sqrt{g} \left(R\, e^{2\,\hat\phi} - 8\, e^{\hat\phi} \Delta e^{\hat\phi} \right) \approx 0 \,, \\ \mathcal{H}^i_{\hat\phi} =& -2\, \nabla_j p^{ij} + \pi\, \nabla^i \phi \approx 0 \,, \qquad \mathcal{Q}_{\hat\phi} = \pi - 4\,(p - \langle p \rangle \sqrt{g}) \approx 0 \,, \end{aligned} \tag{242}$$

where we recognize the Lie derivative of the scalar field in the smeared version of the diffeo constraint $(\mathcal{H}^i_{\hat\phi} | \xi_i) = \int \mathrm{d}^3x \left(\pi^{ij} \pounds_\xi g_{ij} + \pi \pounds_\xi \phi \right)$.

The equations of motion of the Linking Theory

Let's find the equations of motion generated by the total Hamiltonian of the Linking theory,

$$H_{\text{tot}} = (\mathcal{H}|N) + (\mathcal{Q}|\rho) + (\mathcal{H}^i|\xi_i) \,, \tag{243}$$

they are, after application of the constraints,

$$\begin{aligned} \dot g_{ij} =& 4(\langle \rho \rangle - \rho)\, g_{ij} + \pounds_\xi g_{ij} + 2\, \frac{e^{-6\hat\phi} N}{\sqrt{g}} \left(p_{ij} - \tfrac{1}{3} p g_{ij} \right) \\ & - \tfrac{1}{3} \left\langle N\, e^{6\hat\phi} \sqrt{g} \right\rangle \langle p \rangle\, g_{ij} \,, \end{aligned} \tag{244}$$

$$\begin{aligned} \dot p^{ij} =& -4(\langle \rho \rangle - \rho) \left(\sigma^{ij} - \tfrac{1}{6} \sqrt{g} \langle p \rangle g^{ij} \right) + \pounds_\xi p^{ij} + \tfrac{1}{12} e^{6\hat\phi} \sqrt{g}\, \langle p \rangle^2\, N\, g^{ij} \\ & - 2\, N\, e^{-6\hat\phi} \left(\frac{1}{\sqrt{g}} \sigma^i{}_k \sigma^{kj} + \tfrac{1}{3} \langle p \rangle\, \sigma^{ij} \right) + e^{2\,\hat\phi} \sqrt{g} \left(\tfrac{1}{2} R\, N - \Delta N - 2\, N\, \Delta \hat\phi \right) g^{ij} \\ & + e^{2\,\hat\phi} \sqrt{g} \left(\nabla^i \nabla^j N - R^{ij}\, N - 4N\, \nabla^i \hat\phi \nabla^j \hat\phi - 4\, \nabla^{(i} \hat\phi\, \nabla^{j)} N + 2N\, \nabla^i \nabla^j \hat\phi \right) \\ & + \tfrac{1}{3} \langle p \rangle \left(p^{ij} - \tfrac{1}{2} \sqrt{g} \langle p \rangle g^{ij} \right) \left\langle e^{6\hat\phi} N \right\rangle + \tfrac{1}{12} \left\langle \frac{\delta (\mathcal{H}_\phi | N)}{\delta \hat\phi} \right\rangle \left(e^{6\hat\phi} - 1 \right) \sqrt{g}\, g^{ij}(x) \,, \end{aligned} \tag{245}$$

where $\sigma^{ij} = p^{ij} - \frac{1}{3}\, p\, g^{ij}$ is the traceless part of the momenta. The derivation is again left as an exercise.

The GR gauge

General Relativity is obtained by the gauge fixing $\phi \approx 0$, which is second-class only with respect to $\mathcal{Q}_\phi[\phi, \pi; x)$:

$$\{ \phi(x), (\mathcal{Q}_{\hat\phi} | \rho) \} = \rho(x) \,. \tag{246}$$

This fixes $\rho \approx 0$, which in turn eliminates ϕ and π from the theory, giving the ADM constraints.

The SD gauge

Shape Dynamics follows from the gauge fixing $\pi \approx 0$, which has a single non-vanishing Poisson bracket[46]

$$\{ (\mathcal{H}_{\hat\phi} | N), \pi(x) \} = \frac{\delta (\mathcal{H}_{\hat\phi} | N)}{\delta \hat\phi(x)} - \left\langle \frac{\delta (\mathcal{H}_{\hat\phi} | N)}{\delta \hat\phi} \right\rangle\, e^{6\hat\phi(x)} \sqrt{g(x)} \,, \tag{247}$$

[46] The Poisson bracket $\{ (\mathcal{H}^i_{\hat\phi} | \xi_i), \pi(x) \} = 2\, \nabla^i \left[(p - \langle p \rangle \sqrt{g})\, \xi_i \right]$ vanishes on the gauge-fixing surface: if $\pi \approx 0$ then $\mathcal{Q}_{\hat\phi} \approx 0$ implies $p \approx \langle p \rangle \sqrt{g}$.

where (after applying the constraint $\mathcal{H}_\phi \approx 0$)

$$\frac{\delta(\mathcal{H}_{\hat\phi}|N)}{\delta\hat\phi(x)} \approx \sqrt{g}\, e^{\hat\phi}\left[56\, N\, \Delta e^{\hat\phi} + 8\,\Delta(e^{\hat\phi} N) - 2\, N\left(4\, R\, e^{\hat\phi} + e^{5\hat\phi}\langle p\rangle^2\right)\right], \quad (248)$$

Define now the conformal Laplacian $\bigcirc = 8\,\Delta - R$, which is covariant under conformal transformations, in the sense that if $g_{ab} = e^{4\lambda}\bar g_{ab}$, $\bigcirc f = e^{-4\lambda}\bar\bigcirc(e^{\lambda} f)$, so that when it is applied to $(f\, e^{\hat\phi})$ for any scalar f it is invariant under the transformations generated by $\mathcal{Q}_{\hat\phi}$. We can then rewrite the expression above as follows

$$\frac{\delta(\mathcal{H}_{\hat\phi}|N)}{\delta\hat\phi(x)} \approx \sqrt{g}\, e^{\hat\phi}\left[7\, N\, \bigcirc\, e^{\hat\phi} + \bigcirc(e^{\hat\phi} N) - 2\, N\, e^{5\hat\phi}\langle p\rangle^2\right], \quad (249)$$

which is an explicitly conformally-invariant expression. This is the *Lapse Fixing Equation*. It can be solved for N and admits a one-parameter set of solutions, all related by a constant rescaling. Let's call a solution N_{sol}. Then $\mathcal{H}_{\text{gl}} = (\mathcal{H}_{\hat\phi}|N_{\text{sol}})$ is the part of the Hamiltonian constraint that is first-class with respect to the gauge fixing π and therefore survives it.

The general solution of Eq. (247) consists of a linear combination of the two solutions N_1, N_2 of the homogeneous equation (248) plus a particular solution N_0 of the following nonhomogeneous equation:

$$\frac{\delta(\mathcal{H}_{\hat\phi}|N)}{\delta\hat\phi(x)} = e^{6\hat\phi(x)}\sqrt{g(x)}\,. \quad (250)$$

The complete solution is then

$$N_{\text{sol}} = c_1\, N_1 + c_2\, N_2 + w\, N_0\,, \quad (251)$$

where c_1 and c_2 are spatial (but not necessarily temporal) constants, and w is defined as

$$w = \left\langle \frac{\delta(\mathcal{H}_{\hat\phi}|N)}{\delta\hat\phi} \right\rangle. \quad (252)$$

Plugging the above solution into (247), we get

$$\{(\mathcal{H}_{\hat\phi}|N_{\text{sol}}), \pi(x)\} = w\left[1 - \left\langle e^{6\hat\phi}\sqrt{g}\right\rangle\right] e^{6\hat\phi(x)}\sqrt{g(x)}\,. \quad (253)$$

This is a secondary constraint, for which w plays the role of a Lagrange multiplier. We can write this constraint as

$$\mathcal{H}_{\text{gl}} = \int \mathrm{d}^3x\,\sqrt{g}\left(e^{6\hat\phi} - 1\right) \approx 0\,. \quad (254)$$

The above constraint is not trivial: in the constraints of the Linking Theory there is nothing ensuring that $\hat\phi$ is actually volume-preserving. If one were to solve the constraint $\mathcal{H}_{\hat\phi} \approx 0$ for $\hat\phi$, the solution $\hat\phi[g,p;x)$ would not in general be volume-preserving. Since $\mathcal{H}_{\hat\phi} \approx 0$ completely fixes $\hat\phi$, the condition that $\int \mathrm{d}^3x\,\sqrt{g}\left(e^{6\hat\phi} - 1\right) = 0$ must be considered to be an equation for g_{ij}, p^{ij}. We have therefore identified the leftover global constraint. In Eq. (254), $\hat\phi$ must be considered as the solution $\hat\phi[g,p;x)$ of the LY equation (242), and Eq. (254) must be treated as a constraint for the metric and metric momenta. It is obvious that $\mathcal{H}_{\text{gl}}$ commutes with the conformal constraint $\mathcal{Q}_{\hat\phi}$.

So, reducing phase space by integrating away ϕ and π, the final set of constraints we get is

$$\mathcal{H}_{\text{gl}} = \int \mathrm{d}^3x\,\sqrt{g}\left(e^{6\hat\phi[g,p;x)} - 1\right), \quad \mathcal{H}^i = -2\,\nabla_j p^{ij}\,, \quad \mathcal{Q} = 4(p - \langle p\rangle\sqrt{g})\,, \quad (255)$$

where $e^{6\hat\phi[g,p;x)}$ is the solution of the LY equation. We recognise here the volume-preserving conformal constraint $\mathcal{Q}$ together with a conformally-invariant global Hamiltonian constraint $\mathcal{H}_{\text{gl}}$ that generates the evolution and is a nonlocal functional of the dynamical degrees of freedom g_{ij} and p^{ij}. The proof that $\mathcal{H}_{\text{gl}}$ is both conformally and diffeo-invariant is left as an exercise, and it implies that the above system is first-class.

11.1 The degrees of freedom of Shape Dynamics

As I said in Sec. 9, the reduced configuration space of SD can be clearly identified: it is the quotient of $\mathsf{Superspace}$ by volume-preserving conformal transformations, that is $\mathsf{S}\times\mathbb{R}^+$, conformal superspace plus volume. SD has the structure of a theory that satisfies the strong form of the Mach-Poincaré principle on $\mathsf{S}\times\mathbb{R}^+$. It satisfies the strong and not the weak form of this principle because of the reparametrization constraint $\mathcal{H}_{\text{gl}}$: one needs just a point and a *direction* in $\mathsf{S}\times\mathbb{R}^+$: we only need the increment in S, not the one in $\mathbb{R}^+$ to determine the dynamical orbit – thanks to $\mathcal{H}_{\text{gl}}$.

This state of things might look a bit unnatural: there is just a single, global degree of freedom that does not belong to S and yet is necessary for the dynamics. York, noticing this fact, wrote [79]:

> *The picture of dynamics that emerges is of the time-dependent geometry of shape ('transverse modes') interacting with the changing scale of space ('longitudinal mode').*

With 'transverse modes' he referred to the conformally-invariant degrees of freedom. Notice the singular in 'scale of space' and 'longitudinal mode': he didn't refer to the *local* scales $\sqrt{g(x)}$ but to the single, global one V.

But the reparametrization invariance of SD, expressed by the global constraint $\mathcal{H}_{\rm gl}$, suggests that this 'heterogeneous' degree of freedom V is not as physical as the conformally invariant ones living in S. The other ingredient to understand what's going on here is due to N. Ó Murchadha and J. Barbour, who noticed [94] that in the initial value problem, the choice of the initial volume V is purely conventional: it is just a matter of units of lengths. There is an 'accidental' symmetry in the equations of Shape Dynamics which implies that V and the York time τ are not genuine physical degrees of freedom. We call this symmetry *dynamical similarity*, by analogy with a similar symmetry in Newtonian gravity [9, 14]. This symmetry can be expressed as $(p^{ij}_{\rm TT}, \tau) \to (\alpha^4 p^{ij}_{\rm TT}, \alpha^{-2}\tau)$, where $p^{ij}_{\rm TT}$ is the transverse-traceless part of the momenta, α is a constant (in space and time) and all the other dynamical degrees of freedom are left invariant. The transformation I wrote connects a solution in $\mathsf{S} \times \mathbb{R}^+$ to another solution, but if we project these solutions down to S they both project to the same curve. The two solutions are just related by a change of length units, which cannot have physical significance.

So the punchline is: if, justified by the consideration that a single global unit of length is unphysical, we project the solutions down to S, we end up with a theory satisfying the *weak* Poincaré principle in S.

11.2 The solution to the Problem of Time in SD

In [9] J. Barbour, T. Koslowski and I noticed how Shape Dynamics motivates a simple solution of the notorious *problem of time* of Quantum Gravity, mentioned in Appendix A. The problem of time in GR consists of two levels: first there is the problem of *many-fingered time*, discussed at the end of Sec. 7. For each choice of the lapse function $N(x,t)$ one obtains a solution of ADM gravity which is differently represented in $\mathsf{Superspace}$ but corresponds to the same spacetime, just foliated in another way. This is part of the reason why, upon naîve quantization, one obtains a Wheeler–DeWitt equation that is time-independent (static) (see Sec. A.2). This problem is absent in SD because the refoliation ambiguity is removed by the VPCT constraint: SD is compatible with only one particular foliation of spacetime, that with constant trace of p^{ij}. However the theory is still reparametrization-invariant, even within that particular foliation. Reparametrization-invariant theories have vanishing quantum Hamiltonians, and their quantization gives a static wavefunction(-al) which does not evolve. One strategy to circumvent this issue which has attracted interest in the literature is to accept that the Universe is described by a static wavefunctional and claim that our perception of time is the result of two factors: the wavefunctional being peaked around some semiclassical, high quantum number state, and us having access only to partial information about it [95]. Then the result of our measurements are comparisons of expectation values of *partial observables*, and these evolve only with respect to each other, in a limited sense. To understand this, imagine a 2d quantum harmonic oscillator described through a Hamiltonian constraint like those we encountered in Sec. 6. And imagine that the energy constant E in the Hamiltonian constraint $\mathcal{H} = T - V - E \approx 0$ is equal to the energy of some high quantum number state, $\frac{1}{2}\hbar(n_1 + n_2)$ with n_1 and n_2 large enough. Then the solution of the time-independent Schrödinger equation looks like Fig. 18: a volcano-shaped Probability Density Function. One can, within a certain interval, use one of the two oscillators as an 'internal clock' with respect to which the wavefunction of the other oscillator is seen to evolve. This can be kept up as long as the chosen clock evolves monotonically: as soon as its value approaches a turning point where it inverts its motion, this description becomes untenable. Then a 'grasshopper' strategy is adopted, in which one jumps from one internal clock to the other, exploiting the intervals in which they evolve monotonically. This strategy can be called *Tempus Post Quantum*, in the sense that one seeks a physical definition of time in the observables of the Universe *after* quantizing. There are two problems with this: first, nothing ensures that the wavefunction of the Universe will 'oblige', and get into a semiclassical state. One would like to have a mechanism for which this happens.[47] Second, the 'grasshopper'

[47] We think that our approach provides such a mechanism: read below and ref. [15].

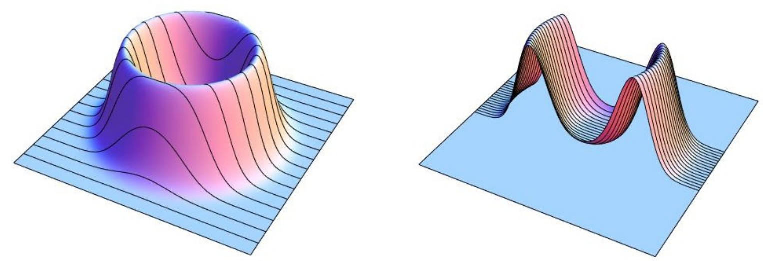

Figure 18: Probability density function for an eigenstate of the 2d quantum harmonic oscillator with sufficiently high quantum numbers. The horizontal axes represent the two oscillator coordinates. One can 'cut' the wavefunction along one axis, and obtain a marginal probability distribution for the other variable. Changing the location of the cut, one obtains a 'time evolution' for the marginal distribution. However, the total probability is not conserved by this evolution. Moreover the evolution cannot be continued indefinitely because, sooner or later, near a 'turning point' for the variable that is being used as internal clock, this interpretation will break down altogether (figure concept: [95]).

strategy is problematic. As soon as one uses an internal clock, even far from the 'turning points', there will be violations of unitarity: one can easily see, for example, that in Fig. 18 the area of the 'cross-section' of the PDF is not conserved.

The strategy we adopted in [9] is the opposite: '*Tempus Ante Quantum*'. If we identify our 'internal clock' before quantizing, we will have a time-dependent Schrödinger equation. This is a desirable situation, but the problem with it is that it is not clear what should provide a good universal internal clock. It should be a quantity that grows monotonically in every solution (no 'turning points') and depends equally on every different part of space, in accordance with some measure). A priori, the search for such a quantity might look as hopeless as Kuchar's search for 'unicorns' [69]. The advance that Shape Dynamics introduces is to single out such variable: I'm talking about $\tau = \frac{2}{3}\langle p\rangle$, the York time. In addition to the good properties for a universal internal time that I already mentioned, York time is geometrically distinguished: it's the only non-shape degree of freedom that plays a dynamical role in SD. τ is clearly monotonic: this can be seen by writing its equations of motion in the Linking Theory and then applying all the constraints and the $\pi \approx 0$ gauge fixing [9]. To use it as an internal clock at the classical level, one has to 'deparametrize' the theory.

Deparametrization: non-autonomous description on S

Deparametrization is a simple idea: take a reparametrization-invariant theory with a Hamiltonian constraint $\mathcal{H}(q_1, p^1, q_2, p^2, \dots) \approx 0$. Say you want to use the variable q_1, in the interval in which it's monotonic, as an internal clock. Then p^1, which is conjugate to q_1 and generates q_1-translations, will play the role of a 'Hamiltonian' that generates evolution in the 'time' q_1. Then we have to solve the Hamiltonian constraint wrt p_1: $p_1 = f(q_1, q_2, p^2, \dots)$ to obtain $\mathcal{H}(q_1, f(q_1, q_2, p^2, \dots), q_2, p^2, \dots) \approx 0$. Then $f(q_1, q_2, p^2, \dots)$ will generate the evolution of all the other variables, $q_2, p^2, \dots$ with respect to q_1:

$$\frac{\mathrm{d}q_i}{\mathrm{d}q_1} = \{f(q_1, q_2, p^2, \dots), q_i\}\,, \qquad \frac{\mathrm{d}p^i}{\mathrm{d}q_1} = \{f(q_1, q_2, p^2, \dots), p^i\}\,, \tag{256}$$

where $i = 2, 3, \dots$.

The variable conjugate to York time is the volume $V = \int \mathrm{d}^3x \sqrt{g}$,

$$\{\tau, V\} = \tfrac{3}{2}\{\langle p\rangle, V\} = \langle\sqrt{g}\rangle = 1\,. \tag{257}$$

Therefore to deparametrize SD with respect to τ, we have to solve the global Hamiltonian constraint for V. The reparametrization constraint of SD is $\mathcal{H}_{\text{gl}}$ in Eq. (255):

$$\mathcal{H}_{\text{gl}} = \int \mathrm{d}^3x\, \sqrt{g}\, e^{6\,\hat\phi[g_{ij}, p^{kl}, \tau; x)} - V\,, \tag{258}$$

where $\hat\phi[g_{ij}, p^{kl}, \tau; x)$ is the solution of the LY equation (242), written in the form

$$\frac{e^{-6\hat\phi}}{\sqrt{g}}\left(p^{ij} - \tfrac{1}{3}pg^{ij}\right)\left(p_{ij} - \tfrac{1}{3}pg_{ij}\right) - \frac{3}{8}\sqrt{g}\, e^{6\hat\phi}\tau^2 - \sqrt{g}\left(R\, e^{2\,\hat\phi} - e^{\hat\phi}\Delta e^{\hat\phi}\right) = 0\,. \tag{259}$$

Now notice that the LY equation is covariant under conformal transformations of the form $g_{ij} \to \lambda g_{ij}$, $p^{ij} \to \lambda\, p^{ij}$, $\hat\phi \to \hat\phi - \log\lambda$ and $\tau \to \tau$, where $\lambda = \lambda(x) > 0$. Therefore $\sqrt{g}\, e^{6\,\hat\phi[g_{ij}, p^{ij}, \tau; x)}$ is fully conformally invariant: it cannot depend on the volume V. Deparametrizing with respect to τ is then immediate: the solution of $\mathcal{H}_{\text{gl}} \approx 0$ for V is simply

$$V = \int \mathrm{d}^3x\, \sqrt{g}\, e^{6\,\hat\phi[g_{ij}, p^{ij}, \tau; x)}\,, \tag{260}$$

and the Hamiltonian generating evolution in τ-time is (see [96], but essentially the same Hamiltonian has been written by York)

$$H_{\text{SD}} = \int \mathrm{d}^3x \sqrt{g}\, e^{6\hat\phi[g_{ij}, p^{kl}, \tau; x)}\,. \tag{261}$$

The above Hamiltonian depends on τ, which we now take as the independent variable. Hamilton's equations are consequently not autonomous (meaning that they depend explicitly on the independent variable) and they are not invariant under τ-translations. This means that among the initial data needed to specify a solution we have to include a value of τ. The initial-value problem is completely specified by local shape initial data (a conformal equivalence class and TT momenta) plus τ. However, one can rewrite this system as an equivalent one which is autonomous, at the cost of having 'friction' terms which make the equations of motion non-Hamiltonian [14]. The key to do this is to do some dimensional analysis: initial data on shape space should be dimensionless, however the metric momenta are dimensionful. One can obtain dimensionless momenta and equations of motion by multiplying p^{ij} by an appropriate power of τ, and reparametrize τ to $\log\tau$, but then the new momenta won't satisfy Hamilton's equations (the difference will just be a dissipative term proportional to p^{ij} in the equation for $\mathrm{d}p^{ij}/\mathrm{d}\log\tau$). Using $\log\tau$ as independent variables allows us only to describe half of each solution: the half in which τ is positive. The other half can be described as a different solution of the same dissipative system. So each solution is split into two half at the instant when $\tau = 0$. This description is suggestive: one can do the same in the Newtonian N-body problem, where the role of τ is played by the dilatational momentum D. The dissipative nature of the equations of motion imply an irreversible growth of a scale-invariant quantity (a function of shape space) which measures the degree of complexity of the shape of the universe. This defines a previously unnoticed arrow of time that points from the simplest and more homogeneous to the more complex and clustered states. In [14] we conjectured that an analogous arrow of time can be identified in geometrodynamics, and may be a better way to think about the evolution of our Universe.

Construction of spacetime

A solution of Shape Dynamics is a curve in conformal superspace, parametrized with York time τ. A way to represent it is witha conformal gauge, for expample the unimodular gauge:

$$\tilde{g}_{ij}(x,t) = \frac{g_{ij}}{\det g^{1/3}} \,, \tag{262}$$

from the tangent vector to the curve $\frac{\mathrm{d}\tilde{g}_{cd}}{\mathrm{d}\tau}$ we can build CMC momenta as

$$\tilde{p}^{ij} = \left(\tilde{g}^{ik}\tilde{g}^{jl} - \tfrac{1}{2}\tilde{g}^{ij}\tilde{g}^{kl}\right)\left(\frac{\mathrm{d}\tilde{g}_{kl}}{\mathrm{d}\tau} + \pounds_\xi \tilde{g}_{kl}\right) + \tfrac{2}{9}\,\tau\,\tilde{g}^{ij} \,, \tag{263}$$

notice that, since $\tilde{g}_{ij}$ is unimodular, $\left(\tilde{g}^{ik}\tilde{g}^{jl} - \frac{1}{2}\tilde{g}^{ij}\tilde{g}^{kl}\right)\left(\frac{\mathrm{d}\tilde{g}_{kl}}{\mathrm{d}\tau} + \pounds_\xi \tilde{g}_{kl}\right)$ is automatically zero-trace. Then we can solve the diffeo constraint for $\xi^i[\tilde{g}_{kl}, \tilde{p}^{kl}; x)$ and make $\tilde{p}^{ij}$ transverse: $\tilde{\nabla}_j \tilde{p}^{ij} = 0$ (as remarked in York, such an equation for ξ^i is elliptic and admits a unique solution on a compact manifold). At this point we have everything that's necessary to solve the Lichnerowicz–York equation and get a scale factor $\phi[\tilde{g}_{ij}, \tilde{p}^{ij}; \tau, x)$ which can be used to define a proper Riemannian 3-metric as $g_{ij} = \phi^4\,\tilde{g}_{ij}$ defining local scales (it is not unimodular). Finally, we can take the last step of solving the Lapse-fixing equation and get a lapse $N[\phi, \tilde{g}_{ij}, \tilde{p}^{ij}; \tau, x)$, with which we can define a 4-dimensional Lorentzian metric:

$$g_{\mu\nu} = \left(\begin{array}{cc} -N^2 + \phi^4\,\tilde{g}_{ab}\,\xi^a\,\xi^b & \phi^4 \tilde{g}_{ac}\,\xi^c \\ \phi^4 \tilde{g}_{bc}\,\xi^c & \phi^4 \tilde{g}_{ab} \end{array} \right) . \tag{264}$$

Notice that nothing ensures that $\det g_{\mu\nu} \neq 0$. The above metric can be degenerate and won't globally define, in general, a spacetime.

The emergence of rods and clocks

In [28] Einstein remarked, about his theory of Relativity:

> *It is striking that the theory (except for four-dimensional space) introduces two kinds of physical things, i.e. (1) measuring rods and clocks, (2) all other things, e.g., the electromagnetic field, material point, etc. This, in a certain sense, is inconsistent; strictly speaking measuring rods and clocks would have to be represented as solutions of the basic equations... not, as it were, as theoretically self-sufficient entities. The procedure justifies itself, however, because it was clear from the very beginning that the postulates of the theory are not strong enough to deduce from them equations for physical events sufficiently complete and sufficiently free from arbitrariness in order to base upon such a foundation a theory of measuring rods and clocks. If one did not wish to forego a physical interpretation of the coordinates in general (something that, in itself, would be possible), it was better to permit such inconsistency - with the obligation, however, of eliminating it at a later stage of the theory.*

The construction of a spacetime metric from a solution of SD I showed before is a first step in the direction sought by Einstein:from scale-invariant and timeless shape-dynamic first principles, one *creates*, as opposed to simply postulating, a structure in which length and duration (proper time) are 'there' to be measured.

But this is only the first step. We must show how the basic equations of the theory lead to the formation of structures which serve as rods and clocks that measure the dynamically created lengths and durations. We know that this happens in the actual Universe. Sufficiently isolated subsystems move along the geodesics of this metric, and the proper time,

$$ds^2 = \left(\phi^4 \tilde{g}_{ij}\,\xi^i \xi^j - N^2\right) d\tau^2 + 2\,\xi_i\,dx^i\,d\tau + \phi^4 \tilde{g}_{ij}\,dx^i\,dx^j \,, \tag{265}$$

turns out to be the time ticked along their worldline by natural clocks belonging to sufficiently isolated and light subsystems, like for example the rotating Earth. Isolated/light subsystems also provide natural rods, for example, rocks on the surface of the Earth, as their sizes can be compared with each other.

The challenge of the "emergence of rods and clocks" program is to prove that, for suitable operational definitions of rods and clocks from material systems(*e.g.* isolated enough galaxies, which have both a characteristic size and a rotation time), they will behave as is simply postulated in the spacetime description. Namely, rods will stay mutually congruent when brought close to each other and compared (apart from Lorentz contraction if they are in relative motion), and clocks will all approximate proper time along their trajectories (which will all be geodesics of the same spacetime metric). In brief, all good rods and clocks will provide mutually consistent data that conspire to form a unique spacetime manifold all observers will agree on.

If this gets well understood, then the cases in which it fails (*e.g.* black holes, the early universe) will become particularly interesting, and we will have a whole new perspective on them that would not be available if the existence of spacetime is taken as the fundamental postulate. The point is that Shape Dynamics is capable of describing situations that cannot be described as a (single, smooth) spacetime manifold, and these are likely to be very relevant for cosmology and astrophysics (not to speak of quantum gravity, where thespacetime 'prejudice' might have severely hampered progress).

12 Asymptotically flat Shape Dynamics

Shape Dynamics makes sense primarily as a description of a spatially closed Universe. However, it gives interesting predictions also if applied to localized subsystems. The simplest case is the asymptotically flat one, in which Σ is open and the *inferred 4-metric* tends to Minkowski's metric at infinity. This is supposed to model a small region of the Universe which is isolated by a vast empty region. The falloff conditions at infinity should capture the effect of the rest of the Universe on the local system: they provide a reference frame. The falloff conditions for the fields that contribute to form the 4-metric are (see Appendix A.4)

$$\begin{aligned} &g_{ij} \to \delta_{ij} + \mathcal{O}(r^{-1}) \,, \quad p^{ij} \to \mathcal{O}(r^{-2}) \,, \\ &N \to 1 + \mathcal{O}(r^{-1}) \,, \qquad \xi^i \to \mathcal{O}(1) \,. \end{aligned} \tag{266}$$

Now Σ is open. A noncompact space does not go along very well with CMC foliations: the volume of space is not a well-defined concept (it is infinite), and therefore the meaning of York time $\frac{2}{3}\langle p \rangle$, which would be its conjugate variable in the compact case, is not clear. A consistent choice is to put $\frac{2}{3}\langle p \rangle$ to zero. This foliation is called *maximal slicing*. It makes sense as an approximation of a localized region of both space and time, in a time interval so short that the expansion of the Universe is negligible.

If $\langle p \rangle = 0$ the canonical transformation that defines the Linking Theory is much simpler: $(g_{ij}, p^{kl}) \to (G_{ij}, P^{kl})$, $(\phi, \pi) \to (\Phi, \Pi)$, where

$$\begin{aligned} &P^{ij} = e^{-4\phi} p^{ij} \,, \qquad G_{ij} = e^{4\phi} g_{ij} \,, \\ &\Pi = \pi - 4 \, g_{ij} \, p^{ij} \,, \quad \Phi = \phi \,. \end{aligned} \tag{267}$$

Now consider the ADM Hamiltonian and diffeo constraints and the $\mathcal{Q}$ constraint for the upper-case variables $(G_{ij}, \Phi, P^{ij}, \Pi)$ and express them in terms of the lowercase ones:

$$\begin{aligned} &\mathcal{H} = \frac{1}{\sqrt{g}} p^{ij} p_{ij} \, e^{-6\phi} - \sqrt{g} \, R \, e^{2\phi} - \sqrt{g} \, e^{\phi} \Delta e^{\phi} \approx 0 \,, \\ &\mathcal{H}^i = -2 \nabla_j \, p^{ij} + \pi \, \nabla^a \phi \approx 0 \,, \qquad \mathcal{Q} = \pi - 4 \, p \approx 0 \,. \end{aligned} \tag{268}$$

The lapse-fixing equation in this case is

$$\sqrt{g} \, e^{-4\phi} \left(\Delta N + 2 \, g^{ij} \, \nabla_i \phi \, \nabla_j N \right) - \tfrac{1}{\sqrt{g}} \, e^{-6\phi} \, N \left(p^{ij} p_{ij} - \tfrac{1}{2} p^2 \right) = 0 \,, \tag{269}$$

while the equations of motion are (we exclude the Hamilton equations for π as they are killed by the SD gauge-fixing $\pi \approx 0$)

$$\begin{aligned} \dot{g}_{ij} &= 4 \, \rho \, g_{ij} + \pounds_\xi g_{ij} + \frac{2}{\sqrt{g}} \, e^{-6\phi} \frac{N}{\sqrt{g}} \, p_{ij} \,, && (270) \\ \dot{\phi} &= \pounds_\xi \phi - \rho \,, && (271) \\ \dot{p}^{ij} &= -4 \, \rho \, p^{ij} + \pounds_\xi p^{ij} - \frac{1}{\sqrt{g}} \, N e^{-6\phi} \left(2 \, p^{ik} p_k{}^j - \tfrac{1}{2} p^{kl} p_{kl} \, g^{ij} \right) && (272) \\ &\quad + \sqrt{g} \, N \, e^{2\phi} \left(R^{ij} - 2 \, \nabla^i \nabla^j \phi + 4 \, \nabla^i \phi \, \nabla^j \phi - \tfrac{1}{2} \, R \, g^{ij} + 2 \, \Delta \phi \, g^{ij} \right) \\ &\quad - \sqrt{g} \, e^{2\phi} \left(\nabla^i \nabla^j N - 4 \, \nabla^{(i} \phi \, \nabla^{j)} N - \Delta N \, g^{ij} \right) \,. \end{aligned}$$

We have to assume some falloff conditions for all the fields in the Linking theory. In order to preserve the asymptotic form of the metric variables and obtain finite boundary terms, we are then forced to take the falloff conditions for the additional degrees of freedom of the Linking Theory to be

$$\rho \to \mathcal{O}(r^{-1}) \,, \qquad e^{\phi} \to 1 + \mathcal{O}(r^{-1}) \,, \qquad \pi \to \mathcal{O}(r^{-2}) \,. \tag{273}$$

Boundary Hamiltonian

As shown in Appendix A.4, asymptotically flat ADM gravity requires a boundary term to be included in the total Hamiltonian. H. Gomes showed in [97] that an analogous boundary Hamiltonian is required in the Linking Theory:

$$E_{\text{SD}} = - \int_{\partial\Sigma} d^2x \, \sqrt{g} \, \left(2 \, k - 2 \, k_0 + 8 \, \partial_r e^{\phi} \right) \,. \tag{274}$$

This Hamiltonian is conformally invariant.

12.1 The wormhole solution

We now turn to the spherically symmetric case. As shown in Ref [4], the assumption of spherical symmetry, together with asymptotically flat boundary conditions, implies that the solution is static and depends only on a single parameter. The proof requires several steps which I will now highlight.

Spherically symmetry implies that the 3-geometry is conformally flat.

A spherically symmetric 3-metric can be written as

$$ds^2 = h(r) \, dr^2 + k(r) \, \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \,, \tag{275}$$

and, if $h(r)$ and $k(r)$ have compatible boundary conditions, one can always reparametrize r so that $h(r) \, dr^2 \to \frac{k(r)}{r^2} dr^2$ and the metric (275) is conformally related (through the conformal factor $k(r)/r^2$) to the flat metric. Another way of proving this is to explicitly calculate the Cotton tensor $C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{4}\left(\nabla_j R\, g_{ik} - \nabla_k R g_{ij}\right)$ of the metric (275) and check that it vanishes identically. Then the metric is locally conformally flat. So we can write $g_{ij} = \lambda^4 \eta_{ij}$, where $\eta_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$ is the flat metric in spherical coordinates and $\lambda = \lambda(r,t)$. Moreover, to be compatible with the assumed boundary conditions, we must have $\lambda \to 1 + \mathcal{O}(r^{-1})$.

If the metric is conformally flat (it is always in the conformal class of the flat metric), the shape momenta must vanish at all times, that is $p^{ij} - \frac{1}{3} p\, g^{ij} = 0$.

The Lichnerowicz–York and Lapse-fixing equations reduce to Poisson's equation.

It is easy to prove that, due to their conformal covariance, can write the Lichnerowicz–York and the lapse-fixing equation as equations for $\Omega = \lambda e^{\phi}$. Their form is

$$\Delta\Omega = \Omega' + \tfrac{2}{r}\Omega' = 0\,, \tag{276}$$

$$\Delta(N\Omega) = (N\Omega)'' + \tfrac{2}{r}(\Omega N)' = 0\,, \tag{277}$$

where $'$ denotes the radial derivative. We have essentially the same equation for Ω and for ΩN. Their solution is immediate:

$$\Omega = a + \tfrac{b}{r}\,, \qquad N = \frac{c + \frac{d}{r}}{a + \frac{b}{r}}\,, \tag{278}$$

this completely fixes the dependence of Ω and N on the radial coordinates, but leaves free their time-dependence in the form of a possible time-dependence of the integration constants a, b, c and d.

The boundary conditions for Ω are of course $\Omega \to 1 + \mathcal{O}(r^{-1})$, so they fix a to 1. Analogously the boundary conditions for N fix c to 1.

The equations of motion for $\dot g_{ij}$ and $\dot\phi$ together reduce to an equation for Ω alone, relating the shift to b and $\dot b$.

Combining (270) and (272) we get

$$\tfrac{d}{dt}\left(e^{4\phi} g_{ij}\right) = \pounds_\xi \left(e^{4\phi} g_{ij}\right) + \frac{2N e^{4\phi} p_{ij}}{\sqrt{\eta} e^{6\phi}}\,, \tag{279}$$

which does not depend on ρ anymore (of course not: we used the conformal gauge freedom to use the flat metric η_{ij} as our reference metric). Now, using $g_{ij} = \lambda^4 \eta_{ij}$ and $\lambda e^{\phi} = \Omega$ we end up with

$$4\,\Omega^3\,\dot\Omega\,\eta_{ij} = \pounds_\xi \left(\Omega^4 \eta_{ij}\right) + \frac{2N\Omega^4 p_{ij}}{\sqrt{\eta}\Omega^6}\,, \tag{280}$$

which does not containt ϕ and λ separately anymore. We can then take the trace wrt η^{ij} of both sides and use the constraint $p \approx 0$:

$$\dot\Omega = \tfrac{1}{12}\Omega^{-3}\eta^{ij}\pounds_\xi\left(\Omega^4\eta_{ij}\right)\,. \tag{281}$$

Now, if ξ^i respects spherical symmetry then its expression in spherical coordinates must be $\xi^i = \delta^i{}_r\,\xi(r,t)$. Then the above equation turns into a differential equation for ξ:

$$\dot b = \tfrac{1}{6}\left(\tfrac{4b}{r} - 2\right)\xi - \tfrac{1}{6}(b + r)\,\xi'\,. \tag{282}$$

whose solution is (k is a new integration constant)

$$\xi(r,t) = \tfrac{r^4}{(b+r)^6}\, k - \tfrac{r\left(b^6 + 9b^5 r + 45 b^4 r^2 - 60 b^3 r^3 \log r - 45 b^2 r^4 - 9 b r^5 - r^6\right)}{6(b+r)^6}\, \dot b\,. \tag{283}$$

The traceless part of $\frac{d}{dt}(e^{4\phi} g_{ij})$ kills ξ and $\dot b$.

Taking the traceless part of Eq.(280) we get

$$\sigma_{ij} = \frac{\sqrt{\eta}\,\Omega^2}{2N}\left[\pounds_\xi\left(\Omega^4\eta_{ij}\right) - \tfrac{1}{3}\eta_{ij}\eta^{kl}\pounds_\xi\left(\Omega^4\eta_{kl}\right)\right] \tag{284}$$

which is computed to be

$$\sigma_{ij} = \frac{r^2 \sin\theta\left(\frac{b}{r} + 1\right)^5}{r - d}\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3r^2} & 0 \\ 0 & 0 & -\frac{1}{3(r^2\sin^2\theta)} \end{pmatrix}(r\,\xi' - \xi)\,, \tag{285}$$

now, if we want to preserve spherical symmetry, the traceless momenta must be zero at all times. This implies $r\,\xi' - \xi = 0$ whose solution is $\xi = q\,r$, where q is another integration constant.

The condition $\xi = q\,r$, combined with (283), implies that $q = k = 0$ (and therefore $\xi = 0$), and that $\dot b = 0$, so that b is constant in time, and so is the conformal factor:

$$\Omega = 1 + \tfrac{b}{r}\,, \qquad \dot b = 0\,. \tag{286}$$

The equations of motion for $\dot p^{ij}$ imply that $d = b$.

Consider now Eq. (272), and impose that $\sigma^{ij} = 0$. The equation can be written as

$$\dot p^{ij} = (d - b)\frac{(r^3\,\sin\theta)}{(b + r)^4}\mathrm{diag}\left\{-2, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right\}\,. \tag{287}$$

If we want to preserve the condition $p^{ij} = 0$, then, we need to have $d = b = const.$ This fixes also the lapse to be constant in time:

$$N = \frac{1-\frac{b}{r}}{1+\frac{b}{r}}\,, \qquad \dot{b} = 0\,. \tag{288}$$

The spacetime line element is that of an isotropic wormhole.

Finally, we can construct a 4-dimensional line element by going to the gauge $\phi = 0$, which selects the 3-metric $g_{ij} = e^{4\phi}\eta_{ij} = \left(1+\frac{b}{r}\right)^4 \eta_{ij}$, and using the lapse N:

$$\begin{aligned} \mathrm{d}s^2 &= -N^2\,\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \\ &= \boxed{-\left(\frac{1-\frac{b}{r}}{1+\frac{b}{r}}\right)^2 \mathrm{d}t^2 + \left(1+\tfrac{b}{r}\right)^4\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right)}\,. \end{aligned} \tag{289}$$

The line element we constructed possesses an inversion symmetry $r \to \frac{b^2}{r}$ around the sphere $r = b$. Outside of this sphere the line element is just that of Schwarzschild spacetime in isotropic coordinates. Inside it is an 'inverted' copy of it. The origin $r = 0$ is not a singularity like in Schwarzschild, it is instead a 'point at infinity' as it takes an infinite amount of proper time for any timelike geodesic to reach it from any other point. For example, take a radial geodesic parametrized by the radius $t = t(r), \theta = 0, \phi = 0$. In a small neighbourhood of $r = 0$ the geodesic equation reads

$$2\,t'(r) + r\,t''(r) + \mathcal{O}(r^2) = 0\,, \tag{290}$$

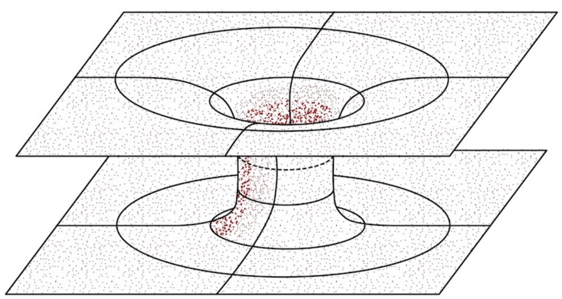

Figure 19: The 3-metric has an inversion symmetry around the sphere $r = b$, and has two asymptotically flat regions. This diagram illustrates the topological structure of the corresponding spatial manifold (suppressing one dimension): two asymptotically flat regions glued together at the dashed line.

and then the solution is, for small r, $t(r) \sim c_1 - c_2/r$ (with $c_2 > b^2$ otherwise the geodesic is not timelike). The proper time elapsed from $r = \epsilon$ to $r = R$ is:

$$\delta\tau = \int_\epsilon^R \mathrm{d}r \sqrt{\left(\frac{1-\frac{b}{r}}{1+\frac{b}{r}}\right)^2 \left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 - \left(1+\frac{b}{r}\right)^4} \sim \int_\epsilon^R \mathrm{d}r\,\frac{\sqrt{c_2 - b^4}}{r^2} \xrightarrow[\epsilon\to 0]{} \infty\,. \tag{291}$$

The above proves that $r = 0$ is an asymptotic region. The 4d Riemann tensor vanishes as $r \to 0$ (left as an exercise), and therefore $r \to 0$ is an asymptotically flat boundary region. What we have is two asymptotically flat regions glued together at an 'event horizon' at $r = b$.

Part V

Appendices

A Arnowitt–Deser–Misner Gravity

A.1 The Arnowitt–Deser–Misner (ADM) formalism

To write GR in Hamiltonian form, we need a notion of time. Therefore assume that spacetime $\mathcal{M}$ is globally hyperbolic, then choose a foliation by spacelike hypersurfaces Σ_t, where t will be a monotonic label for the leaves.

Now consider on $\mathcal{M}$ a system of coordinates *adapted to the foliation*: on each hypersurface Σ_t, introduce some coordinate system (x_1, x_2, x_3). If it varies smoothly between neighboring hypersurfaces, then (x_1, x_2, x_3, t) constitutes a well-behaved coordinate system on $\mathcal{M}$. The theory of foliations tells us that in such a coordinate system the 4-metric ${}^{(4)}g_{\mu\nu}(x,t)$ can be decomposed into the induced metric on the leaves $g_{ij}(x,t)$ plus a scalar $N(x,t)$, called the *lapse*, and a three-vector $N_a : \mathcal{M} \to T(\Sigma_t)$, called the *shift*. Then

$$ {}^{(4)}g_{00} = -N^2 + g_{ij}\, N^i\, N^j \,, \qquad {}^{(4)}g_{0i} = N_i \,, \qquad {}^{(4)}g_{ij} = g_{ij} \,, \tag{292} $$

and the inverse metric ${}^{(4)}g^{\mu\nu}(x,t)$ is

$$ {}^{(4)}g^{00} = -1/N^2 \qquad {}^{(4)}g^{0i} = N^i/N^2 \,, \qquad {}^{(4)}g^{ij} = g^{ij} - N^i\, N^j/N^2 \,. \tag{293} $$

Let $n^\mu(x,t)$ be a unit timelike 4-vector field, $g_{\mu\nu} n^\mu n^\nu = -1$, normal to the three-dimensional hypersurfaces. Its components are

$$ n^\mu = \left(1/N, -N^i/N\right) \,. \tag{294} $$

This equation clarifies the meaning of the lapse N and the shift N^i. The lapse $N(x,t)$ expresses the proper time elapsed between the point $(x,t) \in \Sigma_t$ and the point $(x', t+\delta t)$ on the following infinitesimally close hypersurface $\Sigma_{t+\delta t}$ towards which n^μ points. Starting from the point $(x', t+\delta t)$, one has to move 'horizontally', on the spatial hypersurface, by an amount $N^i(x', t+\delta t)$ to reach the point with coordinates $(x, t+\delta t)$ (see Fig. 20).

To decompose the Einstein–Hilbert action (plus cosmological constant Λ),

$$ S_{\text{EH}} = \int \mathrm{d}^4x\, \sqrt{-{}^{(4)}g}\, \left({}^{(4)}R - 2\,\Lambda\right) \,, \tag{295} $$

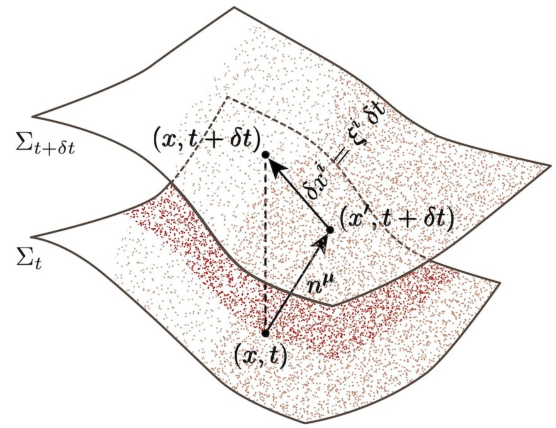

Figure 20: Graphical representation (in 2+1 dimensions) of the meaning of the lapse and the shift.

into its spatial components according to the chosen foliation, we need an expression for the determinant of the 4-metric, which is readily obtained from Eq. (293),

$$ \sqrt{-{}^{(4)}g} = N\,\sqrt{g} \,. \tag{296} $$

We also need the decomposition, named after Gauss and Codazzi (see [32] sec. 8.5a pag. 229), of the 4D Ricci scalar ${}^{(4)}R$ into the 3D intrinsic scalar curvature R and the *extrinsic curvature*,

$$ K_{ij} = \frac{1}{2N}\left(\pounds_{\vec{N}} g_{ij} - \frac{dg_{ij}}{dt}\right) = \frac{1}{2N}\left(\nabla_i N_j + \nabla_j N_i - \frac{\mathrm{d}g_{ij}}{\mathrm{d}t}\right) , \tag{297} $$

of the leaves ($\pounds_{\vec{N}}$ is the Lie derivative wrt the 3-vector field N^i). We get

$$ {}^{(4)}R = R + K_{ij}\,K^{ij} - K^2 - 2\,\nabla_\mu\,(Kn^\mu) - \frac{2}{N}\nabla_i\nabla^i N \,. \tag{298} $$

The Einstein–Hilbert action (295) then reads

$$ \begin{aligned} S_{\text{EH}} = \int \mathrm{d}^4x\,\sqrt{g}\,\{ & N\left(R - 2\,\Lambda + K_{ij}\,K^{ij} - K^2\right) \\ & -2\,N\,\nabla_\mu\,(Kn^\mu) - 2\,\nabla_i\nabla^i N\} \,, \end{aligned} \tag{299} $$

where the next-to-last term is obviously a 4-divergence,

$$ -2\int \mathrm{d}^4x\,\sqrt{g}\,N\,\nabla_\mu\,(Kn^\mu) = -2\int \mathrm{d}^4x\,\sqrt{-{}^{(4)}g}\nabla_\mu\,(Kn^\mu) = 0 \,, \tag{300} $$

while the last term is a 3-divergence:

$$-2\int \mathrm{d}^4x\,\sqrt{g}\,\nabla_i\nabla^i N = -2\int dt \int \mathrm{d}^3x\,\nabla_i\left(\nabla^i N\right) = 0\,. \tag{301}$$

Thus the final form of the 3+1 decomposition of the Einstein–Hilbert action is

$$S_{\text{EH}} = \int \mathrm{d}^4x\,\sqrt{g}\,N\left(R - 2\,\Lambda + K_{ij}\,K^{ij} - K^2\right)\,. \tag{302}$$

Let's write this in the Hamiltonian language. The coordinates are the 3-metric g_{ij}, the lapse N and the shift N_i (the same number of degrees of freedom, 10, as there were in the original 4-metric). The only time derivatives that appear are those of g_{ij}, through the extrinsic curvature (297), and in particular we have no time derivatives of N and N_i, which therefore are just Lagrange multipliers.

The momenta conjugate to g_{ij} are

$$p^{ij} = \frac{\delta\mathcal{L}_{\text{EH}}}{\delta\dot{g}_{ij}} = \sqrt{g}\left(K\,g^{ij} - K^{ij}\right)\,, \tag{303}$$

(notice the $\sqrt{g}$ factor: p^{ij} is a symmetric tensor *density* of weight 1).

The canonical Hamiltonian is given by the Legendre transform $\mathcal{H}_{\text{ADM}} = \int \mathrm{d}^3x\left(p^{ij}\,\dot{g}_{ij} - \mathcal{L}_{\text{EH}}\right)$ and (with some boundary terms discarded) is equivalent to

$$\mathcal{H}_{\text{ADM}} = \int \mathrm{d}^3x\left(N\,\mathcal{H} + N_i\,\mathcal{H}^i\right)\,, \tag{304}$$

where (with the trace $p = g_{ij}p^{ij}$)

$$\mathcal{H} = \frac{1}{\sqrt{g}}\left(p_{ij}p^{ij} - \frac{1}{2}p^2\right) + \sqrt{g}\,(2\,\Lambda - R) \approx 0\,, \tag{305}$$

is called the *Hamiltonian*, or *quadratic* constraint, and

$$\mathcal{H}^i = -2\,\nabla_j p^{ij} \approx 0\,, \tag{306}$$

is the *diffeomorphism*, or *momentum* constraint. The ADM Hamiltonian is a linear combination of constraints with N and N_i playing the role of Lagrange multipliers. It therefore vanishes, and there is no preferred notion of time. This is an expression of the reparametrization invariance of GR, and leads to the 'problem of time'.

Notice the minus sign of $p^2/2$ in the kinetic term $p_{ij}p^{ij} - \frac{1}{2}p^2$ of (305). It is related to the Gauss–Codazzi equations, and has nothing to do with the Lorentzian signature of spacetime. The Lorentzian signature can be read off the sign in front of R, which is negative, and would take the opposite sign if spacetime were Euclidean.

A comment on the interpretation of the constraint (306). I said it generates 3-diffeomorphisms. In fact, if it is smeared with a vector field ξ_i, $(\xi_i|\mathcal{H}^i) = \int \mathrm{d}^3x\,\xi_i(x)\,\mathcal{H}^i(x)$ and the Poisson brackets with the metric are taken,

$$\left\{g_{ij}(x), (\xi_k|\mathcal{H}^k)\right\} = \nabla_i\xi_j + \nabla_j\xi_i = \pounds_\xi g_{ij}\,. \tag{307}$$

then the metric transforms as $g_{ij} \to g_{ij} + \pounds_\xi g_{ij}$. The Lie derivative [32, 98] $\pounds_\xi$ is the way an infinitesimal diffeomorphism like $x'_i = x_i + \xi_i$ acts on tensor fields of any kind. See appendix B.3 for details.

Further reading: ADM's review of their original papers [99], and the more recent review [100]. Misner, Thorne, and Wheeler's *Gravitation* [85], Frankel's book on the geometry of physics [32] and Schutz's *Geometrical Methods of Mathematical Physics* [59].

A.2 The Wheeler–DeWitt equation

The quantization of the ADM representation of Einstein's theory can be understood – only formally – in the language of the Schrödinger functional $\Psi : \mathsf{Riem} \to \mathbb{C}$, where Riem is the space of Riemannian three-metrics. The ADM constraints (305) and (306) translate into operator equations on the wave functional $\Psi[g]$. The one associated with the quadratic constraint is called the Wheeler–DeWitt equation and, ignoring operator ordering issues, is

$$\hat{\mathcal{H}}\,\Psi = 0\,, \qquad \hat{\mathcal{H}} = \frac{1}{g}\left(g_{ik}\,g_{jl} - \frac{1}{2}g_{ij}\,g_{kl}\right)\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g_{kl}} - R + 2\,\Lambda\,. \tag{308}$$

This equation is the functional analogue of a time-independent Schrödinger equation[48] for the Hamiltonian $\hat{\mathcal{H}}$ with eigenvalue 2Λ. The other equation is:

$$\hat{\mathcal{H}}^i\,\Psi = 0\,, \qquad \hat{\mathcal{H}}^i = \nabla_j\frac{\delta}{\delta g_{ij}}\,, \tag{309}$$

which enforces invariance of the wave functional $\Psi[g]$ under diffeomorphisms.

[48] As a differential equation, it is closer to a Klein–Gordon equation, being a hyperbolic functional differential equation (due to the minus sign in the kinetic term), whereas in non-relativistic quantum mechanics the time-independent Schrödinger equation is elliptic.

A.3 The Baierlein–Sharp–Wheeler action

In 1962 Baierlein, Sharp and Wheeler [101] found an action for GR which is of Jacobi type, explicitly enforcing reparametrization invariance. Consider the Einstein–Hilbert action in ADM Lagrangian variables (with the explicit metric velocities $\dot{g}_{ij}$ in place of the extrinsic curvature). The action has an interesting dependence on the lapse N:

$$S_{\text{EH}} = \int \mathrm{d}^3x\, \mathrm{d}t \sqrt{g}\, \left[N(R - 2\Lambda) + \tfrac{1}{4}N^{-1}\,T\right] , \tag{310}$$

where the kinetic term T is

$$T = (g^{ik}g^{jl} - g^{ij}g^{kl}) \left[\frac{dg_{ij}}{dt} - \pounds_{\vec{N}} g_{ij}\right] \left[\frac{dg_{kl}}{dt} - \pounds_{\vec{N}} g_{kl}\right] . \tag{311}$$

Varying the action wrt N,

$$-\tfrac{1}{4}N^{-2}\,T + R - 2\Lambda = 0 , \tag{312}$$

we can solve the resulting equation for N,

$$N = \tfrac{1}{2}\sqrt{\frac{T}{R - 2\Lambda}} , \tag{313}$$

and, substituting last expression in the action, eliminate the lapse from it:

$$S_{\text{BSW}} = \int \mathrm{d}^3x\, dt \sqrt{g}\sqrt{R - 2\Lambda}\sqrt{T} . \tag{314}$$

This is the BSW action. We got this action from ADM, let's show that we can do the converse. The canonical momenta are

$$p^{ij} = \frac{\delta\mathcal{L}}{\delta\dot{g}_{ij}} = \sqrt{\frac{g(R - 2\Lambda)}{T}}(g^{ik}g^{jl} - g^{ij}g^{kl}) \left(\frac{dg_{kl}}{dt} - \pounds_{\vec{N}} g_{kl}\right) . \tag{315}$$

As usual in Jacobi-type actions, there is a primary constraint involving the momenta that comes from the square-root form of the action. In this case, the constraint is just the ADM quadratic constraint (305).

The vector field N_i appears without any time derivative, and is therefore a Lagrange multiplier. This implies a primary constraint stating that the momentum conjugate to N_i vanishes:

$$p^i_N = \frac{\delta\mathcal{L}}{\delta\dot{N}_i} = 0 . \tag{316}$$

From the Euler–Lagrange equations for N_a, we get a secondary constraint, saying that the variation of the action with respect to N_a vanishes. This is the ADM diffeomorphism constraint:

$$\frac{\delta\mathcal{L}}{\delta N_i} = 2\,\nabla_j p^{ij} = 0 . \tag{317}$$

If we calculate the Hamiltonian, through a Legendre transformation, we get

$$\begin{aligned} H_{\text{BSW}} &= \int \mathrm{d}^3x \left(p^{ij}\,\dot{g}_{ij} - \mathcal{L}_{\text{BSW}}\right) \\ &= \int \mathrm{d}^3x \sqrt{g}\,\sqrt{T}\left(\frac{p_{ij}\,p^{ij} - p^2/2}{\sqrt{R - 2\Lambda}} + p^{ij}\,\pounds_{\vec{N}} g_{ij} - \sqrt{(R - 2\Lambda)\,T}\right) . \end{aligned} \tag{318}$$

If now we define $N = \sqrt{\frac{T}{4(R-2\Lambda)}}$ and integrate by parts $p^{ij}\,\pounds_{\vec{N}} g_{ij}$:

$$H_{\text{BSW}} = H_{\text{ADM}} = \int \mathrm{d}^3x \sqrt{g}\left(N\,\mathcal{H} + N_i\,\mathcal{H}^i\right) , \tag{319}$$

we see that the theory is equivalent to GR in the ADM formulation.

A.4 Asymptotically flat ADM

The ADM hamiltonian (304), which I reproduce here:

$$H_{\text{ADM}} = \int_\Sigma \mathrm{d}^3x \left(N\,\mathcal{H} + N_i\,\mathcal{H}^i\right) ,$$

does not generate Einstein's equations if Σ is not compact. In fact one has to take into account the boundary terms I discarded in (300) and (301). To calculate the necessary modifications to the Hamiltonian, we can vary (304) and pay attention to each integration by parts

$$\begin{aligned} \delta H_{\text{ADM}} = &\int_\Sigma \mathrm{d}^3x \left\{A^{ij}\,\delta g_{ij} + B_{ij}\,\delta p^{ij}\right\} \\ &+ \int_\Sigma \mathrm{d}^3x \sqrt{g}\,\nabla_k \left(N^k\,p^{ij}\,\delta g_{ij} - 2\,N^i\,p^{jk}\,\delta g_{ij} - 2\,N_j\,\delta p^{jk}\right) \\ &+ \int_\Sigma \mathrm{d}^3x \sqrt{g}\,\nabla^j \left(\nabla^i N\,\delta g_{ij} - N\,\nabla^i \delta g_{ij}\right) \\ &+ \int_\Sigma \mathrm{d}^3x \sqrt{g}\,\nabla^j \left(N\,g^{kl}\,\nabla_i \delta g_{kl} - \nabla_i N\,g^{kl}\,\delta g_{kl}\right) . \end{aligned} \tag{320}$$

Instead of considering this problem in full generality, I will specialize to the asymptotically flat case. The boundary conditions are these: the 4-metric has to reduce to Schwarzschild at spatial infinity, which in Cartesian coordinates reads

$$ds^2 \xrightarrow[r\to\infty]{} -\left(1-\frac{m}{8\,\pi\,r}\right)dt^2+\left(\delta_{ij}+\frac{m}{8\pi}\frac{x_i x_j}{r^3}\right)dx^i dx^j+O(r^{-2})\,. \tag{321}$$

Therefore the spatial metric in a generic spacelike hypersurface goes to the Euclidean one like $g_{ij}-\delta_{ij}\sim r^{-1}$, and its derivatives $g_{ij,k}\sim r^{-2}$. The lapse and the shift can be read off (292) and go like $N-1\sim r^{-1}$ and $N^i\sim r^{-1}$. Their derivatives will go like $N_{,i}\sim r^{-2}$ and $N^i{}_{,j}\sim r^{-2}$. The momenta, defined as (303) and (297), have to go like $p^{ij}\sim r^{-2}$. Now, considering the boundary terms in (320), the only ones that contribute are those that go like r^{-2}, because the surface integrals go like r^2. The only compatible terms belong to the last two lines, which contribute with the following leading order:

$$\begin{aligned}&\int_{\partial\Sigma} d^2\sigma^i\left(g^{kl}\,\partial_i\delta g_{kl}-\partial^j\delta g_{ij}\right)=-\delta E[g_{ij}]\,,\\ &E[g_{ij}]=\int_{\partial\Sigma} d^2\sigma^j\left(g_0^{ik}\partial_k g_{ij}-g_0^{kl}\partial_j g_{kl}\right)\,,\end{aligned} \tag{322}$$

where g_0^{ij} is the flat metric on the boundary (δ^{ij} in Cartesian coordinates). We have found a local boundary integral that can be added to the ADM Hamiltonian to give a well-defined variational principle:

$$\delta\left(H_{\text{ADM}}+E[g_{ij}]\right)=\int_\Sigma \mathrm{d}^3x\left\{A^{ij}\,\delta g_{ij}+B_{ij}\,\delta p^{ij}\right\}\,. \tag{323}$$

Remarkably, we end up with a generator of the dynamics which is not pure constraint: it is a *true Hamiltonian*, which doesn't vanish on the solutions of the equations of motion. Rather it takes the value $E[g_{ij}]$, which depends on the leading order of the metric at infinity. Moreover, $E[g_{ij}]$ is a conserved quantity: in fact the equations of motion are exactly identical to the compact case where $E[g_{ij}]=0$ (a boundary term does not affect the equations of motion), and therefore $\mathcal{H}_{\text{ADM}}$ alone is conserved and always identical to zero on any solution. Then, since the total Hamiltonian is conserved by definition (because it is time-independent), the boundary term is conserved.

The boundary conditions we have considered for the shift ($N^i\sim r^{-1}$) are too restrictive. In fact the most generic ones are $N^i\sim\xi^i+r^{-1}$ where ξ^i is a vector which is tangential to the boundary and $\xi^i\sim r$ because of Killing vectors at infinity. The new contribution to the variation is

$$\int_{\partial\Sigma} d^2\sigma_k\left(\cancel{\xi^k p^{ij}\delta g_{ij}}-2\xi^i p^{jk}\delta g_{ij}-2\xi_j\delta p^{jk}\right) \tag{324}$$

where the first term vanishes because ξ^i is parallel to the boundary and therefore $d^2\sigma_k\xi^k=0$. The remaining terms can be written as a total variation

$$-\,2\int_{\partial\Sigma} d^2\sigma_k\left(\xi^i\,\delta(p^{jk}\,g_{ij})\right)=-2\,\delta\int_{\partial\Sigma} d^2\sigma_k\left(\xi_j\,p^{jk}\right)\,. \tag{325}$$

So we get the boundary terms

$$B[g_{ij},\xi_k]=2\int_{\partial\Sigma} d^2\sigma_k\left(\xi_j\,p^{jk}\right)\,, \tag{326}$$

which are such that the variational problem is well-posed:

$$\delta\left(\mathcal{H}_{\text{ADM}}+E[g_{ij}]+B[g_{ij},\xi_k]\right)=\int_\Sigma \mathrm{d}^3x\left\{A^{ij}\,\delta g_{ij}+B_{ij}\,\delta p^{ij}\right\}\,. \tag{327}$$

In the pure Schwarzschild case we have $g_{ij}=\delta_{ij}+\frac{m}{8\pi}\frac{x_i x_j}{r^3}$, and therefore $\partial_k g_{ij}=\frac{m}{8\,\pi\,r^3}\left(\delta_{ik}\,x_j+\delta_{jk}\,x_i-3\,\frac{x_i x_j x_k}{r^2}\right)$, $\delta^{ij}\partial_k g_{ij}=-\frac{m\,x_k}{8\pi\,r^3}$ and therefore (since $d\sigma^i$ is parallel to x^i and is normalized so that its integral on a sphere of radius R is $4\,\pi\,R^2$,

$$E_{\text{Schw}}=\frac{m}{4\pi}\int_{S^2} d\cos\theta\,d\phi=m\,, \tag{328}$$

the boundary energy coincides with the Schwarzschild mass.

B Other Appendices

B.1 The case for closed spacelike hypersurfaces

It is well known that on a compact space the total electric charge must be zero. This is a consequence of the Gauss constraint: $\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho$. Defining a region $\Omega\in\Sigma$ with a well-defined boundary $\partial\Omega$, and calling the total charge inside that region $Q(\Omega)=\int_\Omega \mathrm{d}^3x\sqrt{g}\,\rho$, we can prove that $Q(\Omega)=-Q(\Sigma\setminus\Omega)$ which implies that the total charge in Σ, $Q(\Sigma)=Q(\Omega)+Q(\Sigma\setminus\Omega)=0$. The proof makes use of the Gauss law:

$$\left.\begin{aligned}Q(\Omega)&=\int_\Omega \mathrm{d}^3x\sqrt{g}\,\boldsymbol{\nabla}\cdot\boldsymbol{E}=\int_{\partial\Omega}\mathrm{d}\boldsymbol{\sigma}\cdot\boldsymbol{E}\\ Q(\Sigma\setminus\Omega)&=\int_{\Sigma\setminus\Omega}\mathrm{d}^3x\sqrt{g}\,\boldsymbol{\nabla}\cdot\boldsymbol{E}=-\int_{\partial\Omega}\mathrm{d}\boldsymbol{\sigma}\cdot\boldsymbol{E}\end{aligned}\right\}\Rightarrow Q(\Sigma)=0\,. \tag{329}$$

In the case of a 3-metric g_{ab} with a Killing vector ξ_a, defined by the Killing equation $\pounds_\xi g_{ab}=\nabla_a\xi_b+\nabla_b\xi_a=0$, we can prove that an analogous result holds as a consequence of the diffeo constraint:

$$-\,2\nabla_j p^{ij}=j^i\,, \tag{330}$$

where j^a is the contribution to the diffeo constraint due to matter fields. For example $j^i=-\pi\nabla^i\varphi$ for a scalar field and $j^i=E^j\nabla^i A_j-A^i\nabla_j E^j$ for an electromagnetic field. Now, the Killing equation and the diffeo constraint imply that the projection of j^i along the ξ_i direction, $j^i\,\xi_i$, is a divergence:

$$-\,2\nabla_j(p^{ij}\xi_i) = j^i\,\xi_i\,. \tag{331}$$

Therefore, by the same argument for electric charge, $\int_\Sigma \mathrm{d}^3x\, j^i\,\xi_i = 0$ on a closed Σ. Depending on the isometries of g_{ij}, the quantity $j^i\,\xi_i$ might either represent some components of the linear or angular momentum of matter. Therefore, if the metric has isometries, one finds that the Machian constraints (vanishing total linear and angular momentum) arise just as a consequence of the closedness of space.

This argument makes use of Killing vectors, but the generic solution is not guaranteed to possess isometries. This is related to the fact that in the generic case the gravitational field will carry some angular and linear momentum (for example in the form of gravitational waves), and one cannot limit consideration to the matter contribution. The way to properly take into account the gravitational contribution requires a much more subtle treatment which I will not go into here.

Assuming we have an analogous result for the generic case, the consequences of the observation reported here are clear: a closed spatial manifold is Machian, while manifolds with a boundary or open manifolds are subject to boundary conditions that spoil the self-contained nature of the theory. They cannot be descriptions of the whole Universe, because for example they admit the presence of a nonzero angular momentum. They might be at most descriptions of subsystems of the Universe, which do not take into account what is going on outside. In the case of a compact manifold with a boundary this seems pretty obvious, but in the noncompact case it is not. In fact the most popular choice that theoreticians make of the spatial manifold and related boundary conditions is asymptotically flat. This choice is particularly non-Machian, as it requires one to specify the value of an external angular momentum and energy at infinity, which are by necessity externally given and not fixed by the dynamical degrees of freedom inside the Universe. Asymptotically flat spaces are still very useful to describe isolated regions of space, but they shouldn't be used as models for the whole Universe!

B.2 Free-end-point variation

Let S be an action I wish to extremalize over a principal G-bundle, q be the canonical coordinates (the particle coordinates, or the metric and matter fields) and ϕ the compensating coordinates that move us on the fibre,

$$S = \int_{s_1}^{s_2} \mathrm{d}s\,\mathcal{L}(q,\dot q,\phi,\dot\phi)\,. \tag{332}$$

My aim is to extremalize the action given boundary values for the fields q at the endpoints of the trial curve, $q(s_1)$ and $q(s_2)$, but I want to leave the endpoint values of the compensating fields free, so that I actually just specify an initial and final *gauge orbit*. Taking the variation of the action wrt ϕ and $\dot\phi$, I am led to the condition

$$\delta S = \int_{s_1}^{s_2} \mathrm{d}s \left\{ \frac{\delta\mathcal{L}}{\delta\phi} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\delta\mathcal{L}}{\delta\dot\phi}\right)\right\}\delta\phi + \frac{\delta\mathcal{L}}{\delta\dot\phi}\delta\phi\bigg|_{s=s_1}^{s_2}\,. \tag{333}$$

Now, the action has to be stationary with respect to *all* variations $\delta\phi$ around the extremalizing trajectory $\phi(t)$. So it has to be stationary also under fixed-endpoint variations $\{\delta\phi \;\text{ s.t. }\; \delta\phi(s_1) = \delta\phi(s_2) = 0\}$. This implies that the extremalizing trajectory has to satisfy the Euler–Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\phi} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\delta\mathcal{L}}{\delta\dot\phi}\right) = 0\,, \tag{334}$$

but the extremalizing trajectory must also make the action stationary, $\delta S = 0$, and therefore the only possibility is that it is at the same time such that

$$\frac{\delta\mathcal{L}}{\delta\dot\phi}\bigg|_{s=s_1} = \frac{\delta\mathcal{L}}{\delta\dot\phi}\bigg|_{s=s_2} = 0\,. \tag{335}$$

Both boundary terms must vanish because the action has to be stationary with respect to variations with one fixed endpoint and one free endpoint as well.

B.3 Lie derivative

The Lie derivative is a map from any kind of tensor and a vector field to a tensor of the same kind. Here I will only give the definition and some useful properties.

The Lie derivative of a tensor with respect to a vector field ξ^c is the directional derivative in the direction of ξ^c. Associated with ξ there is a vector flow that induces a one-parameter family of diffeomorphisms ϕ_s:

$$\frac{\mathrm{d}\phi_s(x)^i}{\mathrm{d}s} = \xi^i\,.$$

Calling ϕ^*_t the *pullback* [32] of the diffeomorphism on tensor fields, we have that

$$\phi^*_s T^{i_1\dots i_m}_{j_1\dots j_n}(\phi_s(x))$$

is the value of $T^{i_1\dots i_m}_{j_1\dots j_n}$ at $\phi_s(x)$ pulled back to the point x. Then the Lie derivative of $T^{i_1\dots i_m}_{j_1\dots j_n}$ wrt ξ at x is

$$\pounds_\xi T^{i_1\dots i_m}_{j_1\dots j_n} = \lim_{s\to 0}\left(\frac{\phi^*_s T^{i_1\dots i_m}_{j_1\dots j_n}(\phi_s(x)) - T^{i_1\dots i_m}_{j_1\dots j_n}(x)}{s}\right)\,. \tag{336}$$

It is clear then how the Lie derivative represents the action of an infinitesimal diffeomorphism generated by the vector field ξ on an arbitrary tensor field.

Coordinate expression

If $T^{i_1\dots i_m}_{j_1\dots j_n}$ is a m,n-tensor density of weight w, its Lie derivative with respect to a vector field ξ_i is

$$\begin{aligned}\pounds_\xi T^{i_1\dots i_m}_{j_1\dots j_n} &= \xi^k \partial_k T^{i_1\dots i_m}_{j_1\dots j_n} + w\, \partial_k \xi^k\, T^{i_1\dots i_m}_{j_1\dots j_n} \\ &- \partial_k \xi^{i_1}\, T^{k\dots i_m}_{j_1\dots j_n} \dots - \partial_k \xi^{i_m}\, T^{i_1\dots k}_{j_1\dots j_n} \\ &+ \partial_{j_1}\xi^k\, T^{i_1\dots i_m}_{k\dots j_n} + \dots + \partial_{j_n}\xi^k\, T^{i_1\dots i_m}_{j_1\dots k}\,.\end{aligned} \tag{337}$$

Notice that, although not obvious at first sight, the expression on the right-hand side is covariant; in fact we can replace all the partial derivatives ∂_c with covariant derivatives ∇_c, and the Christoffel symbols cancel:

$$\begin{aligned}\pounds_\xi T^{i_1\dots i_m}_{j_1\dots j_n} &= \xi^k \nabla_k T^{i_1\dots i_m}_{j_1\dots j_n} + w\, \nabla_k \xi^k\, T^{i_1\dots i_m}_{j_1\dots j_n} \\ &- \nabla_k \xi^{i_1}\, T^{k\dots i_m}_{j_1\dots j_n} \dots - \nabla_k \xi^{i_m}\, T^{i_1\dots k}_{j_1\dots j_n} \\ &+ \nabla_{j_1}\xi^k\, T^{i_1\dots i_m}_{k\dots j_n} + \dots + \nabla_{j_n}\xi^k\, T^{i_1\dots i_m}_{j_1\dots k}\,.\end{aligned} \tag{338}$$

The Lie derivative obeys the Leibniz rule with respect to both the inner and the tensor product, meaning that

$$\begin{aligned}\pounds_\xi \left(\delta^{\cdots}_{\cdots}\, T^{i_1\dots i_m}_{j_1\dots j_n}\, S^{k_1\dots k_r}_{l_1\dots l_s}\right) &= \delta^{\cdots}_{\cdots}\, \pounds_\xi T^{i_1\dots i_m}_{j_1\dots j_n}\, S^{k_1\dots k_r}_{l_1\dots l_s} \\ &+ \delta^{\cdots}_{\cdots}\, T^{i_1\dots i_m}_{j_1\dots j_n}\, \pounds_\xi S^{k_1\dots k_r}_{l_1\dots l_s}\,,\end{aligned} \tag{339}$$

where $\delta^{\cdots}_{\cdots}$ denotes any combination of Kronecker deltas, as for example $\delta^{k_6}{}_{i_3}\delta^{j_3}{}_{k_2}\delta^{j_7}{}_{k_1}$.

Unlike the covariant derivative, the Lie derivative does not depend on the metric. For this reason, when one takes metric variations of expressions containing the Lie derivative, the variation commutes with it:

$$\frac{\delta}{\delta g_{ij}}\left(T^{i_1\dots i_m}_{j_1\dots j_n}\, \pounds_\xi S^{k_1\dots k_r}_{l_1\dots l_s}\right) = \frac{\delta}{\delta g_{ij}} T^{i_1\dots i_m}_{j_1\dots j_n}\, \pounds_\xi S^{k_1\dots k_r}_{l_1\dots l_s} + T^{i_1\dots i_m}_{j_1\dots j_n}\, \pounds_\xi \left(\frac{\delta}{\delta g_{ij}} S^{k_1\dots k_r}_{l_1\dots l_s}\right)\,.$$

Examples

The Lie derivative of the metric field is

$$\pounds_\xi g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i\,, \tag{340}$$

(note that the right-hand side depends on the metric, but only because the metric appears in the first place on the left-hand side). As can be deduced from the expression (336), the Lie derivative respects also the symmetry of the tensors it acts on. For example, its action on a covariant (meaning with downstairs indices) symmetric tensor like the extrinsic curvature is

$$\pounds_\xi K_{ij} = \xi^k \nabla_k K_{ij} + \nabla_i \xi^k\, K_{kj}\,. \tag{341}$$

If the tensor is contravariant (meaning with upstairs indices) and a tensor density of weight $w = 1$ like the metric momentum, then its Lie derivative is

$$\pounds_\xi p^{ij} = \xi^k \nabla_k p^{ij} + \nabla_k \xi^k\, p^{ij} - \nabla_k \xi^i\, p^{kj} - \nabla_k \xi^j\, p^{ik}\,. \tag{342}$$

B.4 TT decomposition of tensors

Helmholtz Decomposition Theorem

Any vector field $\mathbf{F}$ on a 3-space Σ (which we assume endowed with a Riemannian metric g_{ab}, even though it's not necessary for the theorem) can be written as the sum of a transverse $\mathbf{F}_{\mathrm{T}}$ and longitudinal $\mathbf{F}_{\mathrm{L}}$,

$$\mathbf{F} = \mathbf{F}_{\mathrm{T}} + \mathbf{F}_{\mathrm{L}}\,, \tag{343}$$

where $\boldsymbol{\nabla}\cdot\mathbf{F}_{\mathrm{T}} = 0$ and $\boldsymbol{\nabla}\times\mathbf{F}_{\mathrm{L}} = 0$. The two parts can be written respectively as the curl of a vector field $\boldsymbol{\theta}$ and the divergence of a scalar field ϕ:

$$\mathbf{F}_{\mathrm{T}} = \boldsymbol{\nabla}\times\boldsymbol{\theta}\,, \qquad \mathbf{F}_{\mathrm{L}} = \boldsymbol{\nabla}\phi\,. \tag{344}$$

If Σ is compact, or if it is noncompact but all the fields have appropriate fall-off conditions at infinity, the two parts are mutually orthogonal with respect to the natural global inner product between vector fields (this is where the metric plays a role), as can be proved with an integration by parts:

$$\begin{aligned}(\mathbf{F}_{\mathrm{T}}|\,\mathbf{F}_{\mathrm{L}}) &= \int \mathrm{d}^3x\,\sqrt{g}\;\mathbf{F}_{\mathrm{T}}\cdot\mathbf{F}_{\mathrm{L}} = \int \mathrm{d}^3x\,\sqrt{g}\;\boldsymbol{\nabla}\times\boldsymbol{\theta}\cdot\boldsymbol{\nabla}\phi \\ &= -\int \mathrm{d}^3x\,\sqrt{g}\;\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\theta})\,\phi = 0\,.\end{aligned} \tag{345}$$

The decomposition can be made by solving the equation $\boldsymbol{\nabla}\cdot\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F}_{\mathrm{L}}$ and $\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F}_{\mathrm{T}}$ for ϕ,

$$\Delta\phi = \boldsymbol{\nabla}\cdot\mathbf{F}\,. \tag{346}$$

This is Poisson's equation for ϕ with source $\boldsymbol{\nabla}\cdot\mathbf{F}$, which is well-known to admit a unique solution. Once the solution has been found, let's call it $\nabla^{-2}\boldsymbol{\nabla}\cdot\mathbf{F}$, the transverse part can be readily defined as $\mathbf{F}_{\mathrm{T}} = \mathbf{F} - \boldsymbol{\nabla}(\nabla^{-2}\boldsymbol{\nabla}\cdot\mathbf{F})$.

This decomposition is unique modulo a harmonic part, i.e., a field which solves the Laplace equation

$$\mathbf{F}_{\mathrm{L}} = \boldsymbol{\nabla}\Lambda\,, \qquad \Delta\Lambda = 0\,. \tag{347}$$

This equation admits only the zero solution on closed spaces, and the same holds for simply connected noncompact spaces if vanishing boundary conditions are chosen.

York's conformally covariant decomposition of symmetric tensors

In close analogy to the Helmholtz decomposition theorem, symmetric tensors admit a decomposition into a transverse-traceless (spin-2) part, a longitudinal (spin-1) part, and a pure trace, scalar part.

$$X^{ij} = X^{ij}_{\mathrm{TT}} + X^{ij}_{\mathrm{L}} + X^{ij}_{\mathrm{tr}} = X^{ij}_{\mathrm{TT}} + (LY)^{ij} + \tfrac{1}{3}\,X\,g^{ij}\,, \tag{348}$$

where $X = g_{ij}X^{ij}$, and

$$(LY)^{ij} = \nabla^i Y^j + \nabla^j Y^i - \tfrac{2}{3}\,g^{ij}\,\nabla_k Y^k \tag{349}$$

is the *conformal Killing form* of the vector field Y^i. It can be obtained as the Lie derivative wrt Y^i of the unit-determinant part $g^{-1/3}g_{ij}$ of the metric:

$$(LY)_{ij} = g^{1/3}\,\pounds_Y(g^{-1/3}g_{ij})\,. \tag{350}$$

We can solve for Y^a the transversality condition

$$\nabla_j(LY)^{ij} = \nabla_j(X^{ij} - \tfrac{1}{3}\,X\,g^{ij}), \tag{351}$$

on the left-hand side of which we have a linear second-order differential operator acting on Y^i (the Ricci tensor comes from commuting the covariant derivatives: $[\nabla_j, \nabla^i]Y^j = R^i{}_j Y^j$),

$$\nabla_j(LY)^{ij} = (\Delta_{\mathrm{L}}Y)^i = \Delta Y^i + \tfrac{1}{3}\nabla^i\nabla_j Y^j + R^i{}_j Y^j\,. \tag{352}$$

Here the operator Δ_{L} is strongly elliptic, as can be seen by studying its *principal symbol*, that is, the matrix obtained by replacing each derivative by an arbitrary variable $\partial_i \to z_i$ and taking only the highest derivatives (second):

$$\sigma_{\mathrm{P}}(\Delta_{\mathrm{L}}) = \delta^i{}_j\,z^k z_k + \tfrac{1}{3}z^i z_j\,. \tag{353}$$

If this matrix has positive determinant for any value of z_i, the operator is *elliptic*, and if its eigenvalues are always positive the operator is *strongly elliptic*. Both conditions are realized: $\det\sigma_{\mathrm{P}} = \frac{4}{3}\,(z^k z_k)^3$ and the eigenvalues are $z^k z_k$ with multiplicity 2 and $\frac{4}{3}\,z^k z_k$.

The operator Δ_L also has the property of being Hermitian with respect to the natural global inner product between vectors, as can be seen with two integrations by parts,

$$\begin{aligned}(Z_i|(\Delta_{\mathrm{L}}Y)^i) &= \int \mathrm{d}^3x\,\sqrt{g}\,Z_i\,\nabla_j(LY)^{ij} = -\int \mathrm{d}^3x\,\sqrt{g}\,\nabla_j Z_i(LY)^{ij}\\ &= \int \mathrm{d}^3x\,\sqrt{g}\,Y_i\,\nabla_j(LZ)^{ij} = (Y_i|(\Delta_{\mathrm{L}}Z)^i)\,.\end{aligned} \tag{354}$$

Now let's come to the kernel of Δ_{L}, which represents the analogue of harmonic fields in the Helmholtz decomposition. The equation for the kernel is

$$(\Delta_{\mathrm{L}}\xi)^i = \Delta\xi^i + \tfrac{1}{3}\nabla^i\nabla_j\xi^j + R^i{}_j\xi^j = 0\,. \tag{355}$$

On compact manifolds, or on noncompact manifolds but assuming that Y^a asymptotically approaches zero sufficiently fast, the above equation is equivalent to the vanishing of $(L\xi)^{ij}$,

$$(L\xi)^i = \nabla^i\xi^j + \nabla^j\xi^i - \tfrac{2}{3}\,g^{ij}\,\nabla_k\xi^k = 0\,. \tag{356}$$

This equation identifies *conformal Killing vectors* of the metric g_{ij}, namely vectors that generate infinitesimal diffeomorphisms which leave the metric invariant up to a conformal transformation,

$$\pounds_\xi g_{ij} = \phi\,g_{ij}\,, \qquad \phi = \tfrac{2}{3}\,\nabla_k\xi^k\,. \tag{357}$$

In an asymptotically flat space the conformal Killing vectors will not vanish at infinity and cannot therefore be ignored. One can, however, relax the boundary conditions for the Y^a field, requiring that it approach one of the conformal Killing vectors of Euclidean space. Then Eq. (351) has a unique solution. These boundary conditions are useful for defining the total momentum of the gravitational field in the asymptotically flat case [76]. In the closed case, the good news is that the conformal Killing vectors are always by construction orthogonal (according to the natural global inner product) to the source of Eq. (351), namely the divergence of the traceless part $\nabla_j(X^{ij} - \frac{1}{3}\,X\,g^{ij})$ of X^{ij}:

$$\begin{aligned}(\xi_i|\nabla_j(X^{ij} - \tfrac{1}{3}\,X\,g^{ij})) &= -\int \mathrm{d}^3x\sqrt{g}\,\nabla_j\xi_i(X^{ij} - \tfrac{1}{3}\,X\,g^{ij})\\ &= \int \mathrm{d}^3x\sqrt{g}\,(\nabla_j\xi_i - \tfrac{1}{3}\,\nabla_k\xi^k\,g_{ij})X^{ij}\\ &= \tfrac{1}{2}\int \mathrm{d}^3x\sqrt{g}\,(L\xi)_{ij}X^{ij} = 0\,.\end{aligned} \tag{358}$$

Since the source term in Eq. (351) is orthogonal to the kernel of the operator Δ_{L}, this operator is invertible in the subspace to which $\nabla_j(X^{ij} - \frac{1}{3}\,X\,g^{ij})$

belongs. Equation (351) therefore admits a unique solution modulo conformal Killing vectors, the addition of which does not change the TT-decomposition of X^{ij} because X^{ij}_{L} is insensitive to them.

The three terms in the TT-decomposition are orthogonal to each other:

$$\begin{aligned}(X_{\text{TT}}|X_{\text{L}}) &= -2(\nabla_j X^{ij}_{\text{TT}}|\xi_i) - \tfrac{2}{3}(g_{ij}X^{ij}_{\text{TT}}|\nabla_k\xi_k) = 0\,,\\ (X_{\text{TT}}|X_{\text{tr}}) &= \tfrac{1}{3}(g_{ij}X^{ij}_{\text{TT}}|X) = 0\,,\\ (X_{\text{L}}|X_{\text{tr}}) &= \tfrac{1}{3}(g_{ij}(LY)^{ij}|X) = 0\,.\end{aligned} \tag{359}$$

One could further decompose the longitudinal part in the manner of Helmholtz into a pure-spin one and a pure-scalar part, but this last decomposition is not conformally covariant.

Conformal covariance of the decomposition

Make the conformal transformation

$$\bar g_{ij} = \phi^4 g_{ij}\,, \qquad \bar g^{ij} = \phi^{-4} g^{ij} \tag{360}$$

of the metric and assume that the transformation acts on a symmetric tensor X^{ij} as follows:

$$\bar X^{ij} = \phi^{-10} X^{ij}\,. \tag{361}$$

We will see a posteriori ϕ^{-10} is the only scaling law that leads to conformal covariance for a contravariant symmetric 2-tensor. This can be understood by considering the fact that the metric momenta p^{ij} have to transform in the opposite way to the metric:

$$\bar p^{ij} = \phi^{-4} p^{ij}\,. \tag{362}$$

But p^{ij} is a tensor density, and to have a proper tensor we have to divide it by $\sqrt{g}$, which transforms as ϕ^6. This explains where the ϕ^{-10} factor comes from. More precisely, the exponent of (361) is fixed by the form of (348).

Let's now consider the transformation of the TT-part of X^{ij}. We first recall its definition by (348),

$$X^{ij}_{\text{TT}} = X^{ij} - \tfrac{1}{3}\,X\,g^{ij} + (LY)^{ij} = \phi^{10}(\bar X^{ij} - \tfrac{1}{3}\,\bar X\,\bar g^{ij}) + (LY)^{ij}\,, \tag{363}$$

and we then remember that

$$(LY)^{ij} = g^{ik}g^{jl}g^{1/3}\pounds_Y(g^{-1/3}g_{kl}) = \phi^4 \bar g^{ik}\bar g^{jl}\bar g^{1/3}\pounds_Y(\bar g^{-1/3}\bar g_{kl})\,. \tag{364}$$

Let us next denote by $(\bar LY)^{ij}$ the conformal Killing form calculated with $\bar g_{ij}$,

$$X^{ij}_{\text{TT}} = \phi^{10}(\bar X^{ij} - \tfrac{1}{3}\,\bar X\,\bar g^{ij}) + \phi^4(\bar LY)^{ij}\,. \tag{365}$$

After these preparations, we define the transformation of X^{ij}_{TT} through

$$\bar X^{ij}_{\text{TT}} = \phi^{-10}X^{ij}_{\text{TT}} = (\bar X^{ij} - \tfrac{1}{3}\,\bar X\,\bar g^{ij}) + \phi^{-6}(\bar LY)^{ij}\,. \tag{366}$$

We now show that this tensor is TT with respect to the transformed metric $\bar g_{ij}$. The tracelessness is trivial because every traceless tensor wrt g_{ij} is traceless also wrt $\bar g_{ij}$, and the two terms that comprise X^{ij}_{TT} are separately traceless. The transversality is less obvious. It needs to hold with respect to the transformed covariant derivative $\bar\nabla_i$, which includes the transformed connection

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + 2(\delta^i{}_j\partial_k\log\phi + \delta^i{}_k\partial_j\log\phi - g_{jk}g^{il}\partial_d\log\phi)\,. \tag{367}$$

Let's take the covariant deriative wrt $\bar g_{ij}$ of $\bar X^{ij}_{\text{TT}}$,

$$\begin{aligned}\bar\nabla_j\bar X^{ij}_{\text{TT}} &= \bar\nabla_j(\bar X^{ij} - \tfrac{1}{3}\,\bar X\,\bar g^{ij}) + \bar\nabla_j\left[\phi^{-6}(\bar LY)^{ij}\right]\\ &= \bar\nabla_j\left[\phi^{-10}\left(X^{ij} - \tfrac{1}{3}\,X\,g^{ij} + (LY)^{ij}\right)\right] = \bar\nabla_j(\phi^{-10}W^{ij})\\ &= \phi^{-10}\left(\nabla_j W^{ij} - 10\,W^{ij}\,\partial_j\log\phi + \Delta\Gamma^i_{kj}\,W^{kj} + \Delta\Gamma^j_{jk}\,W^{ik}\right)\,,\end{aligned} \tag{368}$$

where I called $W^{ij} = X^{ij} - \frac{1}{3}\,X\,g^{ij} + (LY)^{ij}$ and $\Delta\Gamma^i_{jk} = 2(\delta^i{}_j\nabla_k\log\phi + \delta^i{}_k\nabla_j\log\phi - g_{jk}\nabla^i\log\phi)$. An explicit calculation shows immediately that

$$10\,W^{ij}\,\partial_j\log\phi - \Delta\Gamma^i_{kj}\,W^{kj} - \Delta\Gamma^j_{jk}\,W^{ik} = 2\,g_{jk}W^{jk}\,\partial^i\log\phi\,, \tag{369}$$

but W^{ij} is traceless and the above expression vanishes. Thus, we have proved that

$$\bar\nabla_j\bar X^{ij}_{\text{TT}} = \phi^{-10}\left[\nabla_j(X^{ij} - \tfrac{1}{3}\,X\,g^{ij}) + \nabla_j(LY)^{ij}\right]\,, \tag{370}$$

and if X^{ij}_{TT} was TT wrt g_{ij} then $\nabla_j(X^{ij} - \frac{1}{3}\,X\,g^{ij}) + \nabla_j(LY)^{ij} = 0$ and $\bar\nabla_j\bar X^{ij}_{\text{TT}} = 0$, *that is, $\bar X^{ij}_{TT}$ is TT with respect to $\bar g_{ij}$.*

It is easy to see that the above statement implies also its converse because the original metric can be obtained from the barred one through the inverse conformal transformation $g_{ij} = \phi^4_{\text{inv}}\,\bar g_{ij} = \phi^{-4}\,\bar g_{ij}$. This exists because, by definition, $\phi \neq 0$, and therefore the whole argument can be used to show that if $\bar X^{ij}_{\text{TT}}$ is TT wrt $\bar g_{ij}$ thenX^{ij}_{TT} is TT wrt g_{ij}.

We conclude that, given a symmetric 2-tensor X^{ij} on a manifold Σ equipped with the metric g_{ij}, it can be decomposed as (348). On a conformally related manifold Σ with the metric $\bar g_{ij} = \phi^4 g_{ij}$, the tensor $\bar X^{ij} = \phi^{-10}X^{ij}$ decomposes in the same way:

$$\bar X^{ij} = \bar X^{ij}_{\text{TT}} + \bar X^{ij}_{\text{L}} + \bar X^{ij}_{\text{tr}}\,, \tag{371}$$

where $\bar X^{ij}_{\text{TT}} = \phi^{-10}X^{ij}_{\text{TT}}$, $\bar X^{ij}_{\text{L}} = \phi^{-10}X^{ij}_{\text{L}}$ and $\bar X^{ij}_{\text{tr}} = \phi^{-10}X^{ij}_{\text{tr}}$, with the vector Y^i that determines the longitudinal part being the same for $\bar X^{ij}_{\text{L}}$ and X^{ij}_{L}.

List of Figures

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